



THE UNIVERSITY OF WOLLONGONG

A GENERALIZED THEORY OF  
OPERATIONAL CALCULUS

by

M.J. Warmus

Preprint No. 6/87

DEPARTMENT OF MATHEMATICS

P.O. BOX 1144, WOLLONGONG, N.S.W. 2500, AUSTRALIA

Dar p. Terezy Simińskaiej

16.11.2015r.

46-01

46-520

## Table of contents

	Page
I. Introduction .....	1
II. The ring $S$ .....	5
III. The ring $F$ .....	14
IV. Distributions and congruent functions .....	21
V. Examples of applications .....	40
VI. The subring $B$ .....	45
VII. The $D$ -Derivative .....	62
VIII. Further applications .....	76
IX. The distribution rings $D$ and $Dg$ .....	103
X. The quotient field $\mathcal{Q}$ .....	108
XI. A theory corresponding to the Laplace-Transformation .....	126
XII. Final considerations .....	129
 References .....	134



sygn. 12 20493  
nr inw. 20493

## I. Introduction

In 1971 I published a paper [1], suggesting a new theory of operational calculus, more general than existing ones. That new theory included the theory of the Laplace-Carson transformation

$$(1.1) \quad g(p) = p \int_0^\infty e^{-pu} f(u) du ,$$

which is an unessential but more convenient modification of the Laplace transformation

$$(1.2) \quad h(s) = \int_0^\infty e^{-su} f(u) du ,$$

and also, in the sense of an isomorphism (see Theorem 28 in [1]), the theory of Mikusinski [2], [3].

It enabled - like the theory of Mikusinski - the introduction of all functions  $f(t)$  integrable in the Lebesgue sense on every finite interval  $a \leq t \leq b$  but, moreover, the use of all methods and theorems from the theory of the Laplace-Carson transformation (or the Laplace transformation) in a smaller class of functions transformable in the Laplace sense, which was not possible in the theory of Mikusinski.

In this paper I propose a generalization of my previous theory, which enlarges the class of feasible functions, enables obtaining some unexpected new theorems and simplifies proofs of some old ones. Moreover, some other theories of generalized functions may be embedded in this latest theory. The main advantages of my previous theory, namely :

1<sup>o</sup> the class of feasible functions  $f(t)$  is larger than the class of functions transformable in the Laplace sense,

2<sup>o</sup> the class of feasible functions  $g(p)$  is larger than the corresponding class of Mikusinski's operators,

3<sup>o</sup> there exists the possibility of calculating with functions of two variables  $F(p, t)$  and thus introducing new techniques for solving differential equations,

4<sup>o</sup> complicated signs of transforms are replaced by the sign of equivalence  $\equiv$ ,

are also the main advantages of the new theory, which, moreover, as a more general one, gives the possibility of introducing more new techniques.

The principal idea is the same as in my previous theory. We divide a ring of feasible functions  $F(p, t)$  into residue classes of equivalent functions in such a way that :

1º the functions

$$F(p, t) \quad \text{and} \quad p \int_0^t F(p, u) du$$

are equivalent,

2º a continuous function  $f(t)$  is equivalent to zero iff it equals zero identically.

The first condition is an extension of the well-known fact that

$$\{f(t)\} \quad \text{and} \quad p \left\{ \int_0^t f(u) du \right\}$$

are equal in Operational Calculus, where  $\{f(t)\}$  means the Laplace-Carson transform of the function  $f(t)$  or the corresponding operator in the theory of Mikusinski, in both cases  $p$  being the operator of differentiation. The second condition is necessary to enable a pass from a solution given in the form of an equivalence to a solution given as an equality.

It follows that, for every feasible function  $F(p, t)$ , the function

$$(1.3) \quad Z(p, t) = F(p, t) - p \int_0^t F(p, u) du$$

must be equivalent to zero. But we have (1.3) iff

$$(1.4) \quad F(p, t) = Z(p, t) + e^{pt} \times Z(p, t),$$

where  $\times$  denotes the convolution defined by the formula

$$(1.5) \quad F_1(p, t) \times F_2(p, t) = p \int_0^t F_1(p, t-u) F_2(p, u) du.$$

It follows from (1.3) and (1.4) that

$$(1.6) \quad e^{pt} \times Z(p, t) = p \int_0^t F(p, u) du .$$

Let us notice that for  $F(p, t) = e^{pt}$  (1.3) gives  $Z(p, t) = 1$ . This means that  $e^{pt}$  cannot be a feasible function. On the other hand, we know from operational calculus that functions of the form  $e^{kp}$ , where  $k$  is an arbitrary real number, must be feasible. This observation gives us an important indication how to construct the set of feasible functions. It shows also how little freedom we have when defining such a set and how it is difficult to find a new definition enlarging this set.

It follows from (1.3) and (1.4) that every function  $Z(p, t)$ , for which function (1.6) is feasible, must be equivalent to zero. My previous theory satisfied this condition. But the optimal definition ought to be the following one :

" A function  $Z(p, t)$  is equivalent to zero iff function (1.6) is feasible ",

because this would give the smallest set of such functions and, in consequence, the largest set of residue classes. The theory presented in this paper uses the above optimal definition. This has also been the reason why I have been seeking after a better, that is, a more natural theory.

If we want to find a function  $g(p)$  equivalent to a given function  $\delta(p, t)$  then the function

$$Z(p, t) = g(p) - \delta(p, t)$$

must be equivalent to zero and (1.4) becomes equivalent to

$$(1.7) \quad e^{-pt} F(p, t) = g(p) - e^{-pt} \delta(p, t) - p \int_0^t e^{-pu} \delta(p, u) du .$$

If we defined the set of feasible functions as the set of functions  $F(p, t)$  satisfying the condition

$$\lim_{t \rightarrow \infty} e^{-pt} F(p, t) = 0 \quad \text{for sufficiently great } p ,$$

and if  $\delta(p, t)$  were such a function, satisfying, moreover, the condition that the integral

$$\int_0^\infty e^{-pu} \delta(p, u) du$$

converges for sufficiently great  $\rho$ , then we would obtain from (1.7)

$$(1.8) \quad g(p) = \rho \int_0^\infty e^{-\rho u} g(p, u) du, \quad \text{for large } \rho.$$

In particular, for  $g(p, t) = f(t)$ , we would obtain (1.1). This way we have discovered the Laplace-Carson transformation once more. We have also shown that this transformation may be extended to functions of two variables  $g(p, t)$ , because (1.8) is a generalization of (1.1).

In this paper we shall deal with a much larger set of feasible functions, preserving the possibility of the Laplace-Carson transformation for a suitable subset of such functions.

---

If an expression  $H(p, t)$  defines a function  $F(p, t)$  then we shall write

$$F(p, t) := H(p, t) .$$

The sign  $\bullet$  will denote the end of a definition, of a proof, of a remark or an example.

## II. The Ring $\mathcal{S}$

(2.1) Definition. The class  $\mathcal{S}$  is the set of all functions, real or complex, of a real or complex variable  $p$  and a real variable  $t$ , such that for every  $F(p, t) \in \mathcal{S}$  there exists such a real non-negative number  $p_0$  that for every natural number  $n \geq p_0$  the function  $f_n(t) := F(n, t)$  is defined for almost every  $t \geq 0$  and integrable in the Lebesgue sense on every finite interval  $0 \leq t \leq T$ . •

The above definition enables us to simplify the presentation, as follows:

(2.2) Notation. We shall write, for  $F, G \in \mathcal{S}$ ,

$$(2.3) \quad F(p, t) = G(p, t)$$

iff there exists a real non-negative number  $p_0$  such that for every natural number  $n \geq p_0$  the functions  $f_n(t) := F(n, t)$  and  $g_n(t) := G(n, t)$  are identical for  $t \geq 0$ . It follows that any equality of the form

$$(2.4) \quad f(t) = g(t)$$

will mean identity of the functions  $f(t)$  and  $g(t)$  for  $t \geq 0$ . In particular,

$$(2.5) \quad f(t) = 0$$

will mean that the function  $f(t)$  equals zero for all  $t \geq 0$ . Therefore the fact that a function  $f(t)$  vanishes for some  $t \geq 0$  only, must be written in another way. •

For example, we shall write

$$\sin np = 0$$

because we have  $\sin nx = 0$  for  $n = 1, 2, \dots$ , although for  $p \neq n$  we have  $\sin np \neq 0$ .

(2.6) Notation. We shall write

$$(2.7) \quad F(p, t) \neq g(p, t)$$

iff (2.3) is not true, that is, iff for every real non-negative number  $p_0$  there exists a natural number  $n \geq p_0$  such that the functions  $f_n(t) := F(n, t)$  and  $g_n(t) := g(n, t)$  are not identical for  $t \geq 0$ . It follows that

$$(2.8) \quad f(t) \neq g(t)$$

will mean that the functions  $f(t)$  and  $g(t)$  are not identical for  $t \geq 0$ . In particular,

$$(2.9) \quad f(t) \neq 0$$

will mean that the function  $f(t)$  does not vanish identically for  $t \geq 0$  (but, may be, vanishes for some  $t \geq 0$ ). The fact that a function  $f(t)$  does not vanish for any  $t \geq 0$  must be written in another way. •

(2.10) Notation. We shall write

$$(2.11) \quad F(p, t) = g(p, t)$$

iff there exists a real non-negative number  $p_0$  such that for every natural number  $n \geq p_0$  the functions  $f_n(t) := F(n, t)$  and  $g_n(t) := g(n, t)$  are equal for almost every  $t \geq 0$ . •

(2.12) Notation. We shall write

$$(2.13) \quad F(p, t) \rightarrow f(t) \quad \text{or} \quad \lim_{p \rightarrow \infty} F(p, t) = f(t)$$

iff  $F(p, t) \in S$ ,  $f(t) \in S$  and, for every  $t \geq 0$ ,

$$(2.14) \quad f_n(t) \xrightarrow[n \rightarrow \infty]{} f(t)$$

where  $f_n(t) := F(n, t)$  for sufficiently great  $n$ .

Similarly, we shall write

$$(2.15) \quad \begin{matrix} \text{a.e.} \\ F(p, t) \end{matrix} \xrightarrow[p \rightarrow \infty]{} f(t) \quad \text{or} \quad \lim_{p \rightarrow \infty} F(p, t) = f(t)$$

iff  $F(p, t) \in S$ ,  $f(t) \in S$  and for almost every  $t \geq 0$  we have (2.14).

We shall also write

$$(2.16) \quad F(p, t) \xrightarrow[p \rightarrow \infty]{} f(t) \quad \text{or} \quad \lim_{p \rightarrow \infty} F(p, t) = f(t)$$

iff  $F(p, t) \in S$ ,  $f(t) \in S$  and  $f_n(t) := F(n, t)$  converges to  $f(t)$  uniformly in every finite interval  $0 \leq t \leq T$ . •

(2.17) Definition. The class  $\mathcal{C}$  is the set of all functions  $f(t)$ , real or complex, of a real variable  $t$ , continuous in every finite interval  $0 \leq t \leq T$ . •

(2.18) Definition. The convolution of functions  $F_1(p, t), F_2(p, t) \in S$  is the function

$$(2.19) \quad F_1(p, t) * F_2(p, t) := p \int_0^t F_1(p, t-u) F_2(p, u) du .$$

(2.20) Theorem. If  $F_1(p, t), F_2(p, t) \in S$  then  $F_1(p, t) * F_2(p, t) \in S$ .

Proof. Let  $p_j$  be a real non-negative number such that for every natural  $n \geq p_j$  the function  $F_j(n, t)$ ,  $j=1,2$ , is defined for almost every  $t \geq 0$  and integrable in the Lebesgue sense on every finite interval  $0 \leq t \leq T$ . Then for every natural number  $n \geq \max(p_1, p_2)$  the function  $F_1(n, t) * F_2(n, t)$  is defined for almost every  $t \geq 0$  and integrable in the Lebesgue sense on every finite interval  $0 \leq t \leq T$ . This means that  $F_1(p, t) * F_2(p, t) \in S$ . •

(2.21) Theorem. If  $F_1(p, t), F_2(p, t) \in S$  then  $F_1(p, t) \times F_2(p, t) \in S$ .

Proof. In the previous notation, for every natural number  $n \geq \max(p_1, p_2)$  the functions  $F_1(n, t)$  and  $F_2(n, t)$  are defined for almost every  $t \geq 0$  and integrable in the Lebesgue sense on every finite interval  $0 \leq t \leq T$ . Then, by virtue of the Convolution Theorem (see [4], Vol.1, p.110), for every natural number  $n \geq \max(p_1, p_2)$  the integral

$$n \int_0^t F_1(n, t-u) F_2(n, u) du$$

exists for almost every  $t \geq 0$  and is integrable in the Lebesgue sense on every finite interval  $0 \leq t \leq T$ . This means that  $F_1(p, t) \times F_2(p, t) \in S$ . •

(2.22) Theorem. The convolution (2.19) is associative, commutative and distributive with respect to addition.

Proof. Let  $F(p, t), G(p, t), H(p, t) \in S$ . Then we have

$$\begin{aligned} & (F(p, t) \times G(p, t)) \times H(p, t) = \rho^2 \int_0^t \int_0^{t-u} F(p, t-u-v) G(p, v) dv H(p, u) du = \\ & = \rho^2 \int_0^t \int_u^t F(p, t-\sigma) G(p, \sigma-u) H(p, u) d\sigma du = \rho \int_0^t F(p, t-\sigma) \rho \int_0^\sigma G(p, \sigma-u) H(p, u) du d\sigma = \\ & = F(p, t) \times (G(p, t) \times H(p, t)). \end{aligned}$$

This means that the convolution (2.19) is associative. We have further

$$F(p, t) \times G(p, t) = \rho \int_0^t F(p, t-u) G(p, u) du = \rho \int_0^t G(p, t-v) F(p, v) dv = G(p, t) \times F(p, t).$$

This means that the convolution (2.19) is commutative. Finally, we have

$$\begin{aligned} F(p, t) \times (G(p, t) + H(p, t)) &= p \int_0^t F(p, t-u) (G(p, u) + H(p, u)) du = \\ &= p \int_0^t F(p, t-u) G(p, u) du + p \int_0^t F(p, t-u) H(p, u) du = \\ &= F(p, t) \times G(p, t) + F(p, t) \times H(p, t), \end{aligned}$$

which means that the convolution (2.19) is distributive with respect to addition. •

(2.23) Theorem. The class  $S$  forms a commutative ring with respect to addition and the convolution (2.19).

Proof. The theorem follows from Theorems (2.20), (2.21) and (2.22). •

(2.24) Remark. If  $F(t) := f(t)$  and  $F(p, t) \in S$ , we write simply  $f(t) \in S$ . Similarly, if  $F(t) := g(t)$  and  $F(p, t) \in S$ , we write simply  $g(t) \in S$ . •

(2.25) Theorem. If  $f(t)$  is a function defined for almost every  $t \geq 0$  and integrable in the Lebesgue sense on every finite interval  $0 \leq t \leq T$  then  $f(t) \in S$ . In particular, if  $f(t) \in \mathcal{C}$  then  $f(t) \in S$ , which means that  $\mathcal{C} \subset S$ .

Proof. The theorem follows immediately from Definition (2.1). •

(2.26) Theorem. The class  $\mathcal{L}$  forms a ring with respect to addition and the multiplication defined as follows

$$f(t) \otimes g(t) := 1/p \int_0^t f(u) g(t-u) du.$$

Proof. The theorem follows from the well-known fact that  $f(t), g(t) \in \mathcal{L}$  imply  $f(t) \pm g(t) \in \mathcal{L}$  and  $f(t) \otimes g(t) \in \mathcal{L}$  (see, for example, [2], [3] or [4]). •

(2.27) Theorem. The ring  $\mathcal{L}$  has no zero divisors, that is, if  $f(t), g(t) \in \mathcal{L}$  and  $1/p \int_0^t f(u) g(t-u) du = 0$  then either  $f(t) = 0$  or  $g(t) = 0$ .

Proof. The proof is to be found in [5], p.16-20, or in [3]. •

(2.28) Theorem. If  $g(n)$  is a sequence defined for all natural numbers  $n \geq p_0 \geq 0$  then  $g(p) \in S$ .

Proof. The theorem follows immediately from Definition (2.1). •

(2.29) Theorem. If  $g(p) \in S$  and  $F(p, t) \in S$  then  $g(p) F(p, t) \in S$ .

Proof. The theorem follows immediately from Definition (2.1). •

(2.30) Theorem. If  $F(p, t) \in S$  and  $f(t)$  is a measurable function bounded on every finite interval  $0 \leq t \leq T$  then  $f(t) F(p, t) \in S$ .

Proof. The theorem follows from Definition (2.1). •

(2.31) Theorem. We have

$$F(p, t) \in S \quad \text{iff} \quad \int_0^t F(p, u) du \in S \quad \text{iff} \quad p \int_0^t F(p, u) du \in S.$$

Proof. If  $F(p, t) \in S$  then for every natural number  $n \geq p_0 \geq 0$  the function

$$\int_0^t F(n, u) du$$

is defined for every  $t \geq 0$  and, being continuous in every finite interval  $0 \leq t \leq T$ , is integrable in the Lebesgue sense on that interval, which means that

$$(2.32) \quad \int_0^t F(p, u) du \in S.$$

Conversely, if we have (2.32) then for every natural number  $n \geq p_0 \geq 0$  the integral

$$\int_0^T F(n, u) du$$

is defined for every  $T \geq 0$ , which means that  $F(n, t)$  is defined for almost every  $t \geq 0$  and integrable in the Lebesgue sense on every finite interval  $0 \leq t \leq T$ , that is,  $F(p, t) \in S$ . Thus we have  $F(p, t) \in S$  iff

$$\int_0^t F(p, u) du \in S.$$

Since, according to Definition (2.1),

$$\int_0^t F(p, u) du \in S \quad \text{iff} \quad p \int_0^t F(p, u) du \in S,$$

the proof is complete. •

(2.33) Theorem. If for every natural number  $n \geq p_0 \geq 0$  functions  $F(n, t)$  and  $G(n, t)$  are continuous in every finite interval  $0 \leq t \leq T$ , which implies  $F(p, t), G(p, t) \in S$ , and if

$$(2.34) \quad F(p, t) \underset{p \rightarrow \infty}{\Rightarrow} f(t), \quad G(p, t) \underset{p \rightarrow \infty}{\Rightarrow} g(t),$$

then  $f(t), g(t) \in S$  and

$$(2.35) \quad \lim_{p \rightarrow \infty} F(p, t) \times G(p, t) \Rightarrow \lim_{p \rightarrow \infty} f(t) \times g(t).$$

Proof. It follows from (2.34) that the functions  $f(t)$  and  $g(t)$  are continuous in every finite interval  $0 \leq t \leq T$  and, therefore, integrable in the Lebesgue sense on that interval. According to Theorem (2.25), we have  $f(t), g(t) \in S$ .

From (2.34) we obtain for  $0 \leq t \leq T$

$$|F(n, t) - f(t)| < \epsilon(n) \quad \text{and} \quad |G(n, t) - g(t)| < \eta(n),$$

where

$$\epsilon(n) \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{and} \quad \eta(n) \xrightarrow[n \rightarrow \infty]{} 0.$$

Consequently, we have

$$\begin{aligned} & | \frac{1}{n} F(n, t) \times G(n, t) - \frac{1}{n} f(t) \times g(t) | = \\ &= | \int_0^t F(n, t-u) G(n, u) du - \int_0^t f(t-u) g(u) du | \leq \\ &\leq \int_0^t |F(n, t-u) G(n, u) - f(t-u) g(u)| du = \\ &= \int_0^t |F(n, t-u) G(n, u) - f(t-u) G(n, u) + f(t-u) G(n, u) - f(t-u) g(u)| du \leq \\ &\leq \int_0^t |F(n, t-u) - f(t-u)| |G(n, u)| du + \int_0^t |f(t-u)| |G(n, u) - g(u)| du \leq \\ &\leq \int_0^T \epsilon(n) (\alpha + \eta(n)) du + \int_0^T P \eta(n) du = \\ &= T Q \epsilon(n) + T P \eta(n) + T \epsilon(n) \eta(n) \xrightarrow[n \rightarrow \infty]{} 0, \end{aligned}$$

where  $|f(t)| \leq P$ ,  $|g(t)| \leq Q$  for  $0 \leq t \leq T$ . This means (2.35). •

(2.36) **Theorem.** If  $F(p, t) \in S$ ,  $r(t) \in C$ ,  $r(t) \neq 0$ , then

$$(2.37) \quad r(t) \times F(p, t) = 0 \quad \text{iff} \quad F(p, t) = 0.$$

a.e.

Proof. If  $f(t) \times F(p, t) = 0$  then also  $1/p^2 \times f(t) \times F(p, t) = 0$ , that is,

$$\frac{1}{p} f(t) \times \int_0^t F(p, u) du = 0.$$

By virtue of Theorem (2.27), there exists such a real  $p_0 \neq 0$  that for every natural  $n \geq p_0$  we have

$$\int_0^t F(n, u) du = 0.$$

a.e.

It follows that  $F(p, t) = 0$ .

a.e.

Conversely, if  $F(p, t) = 0$  then

$$f(t) \times F(p, t) = p \int_0^t f(t-u) F(p, u) du = 0.$$

This completes the proof. •

### III. The Ring $F$

(3.1) Definition. A function  $F(p, t)$  will be called **feasible** iff :

1º  $F(p, t) \in S$ ,

2º there exist real numbers  $a, b$ ,  $0 \leq a < t$ ,  $b \geq 0$  and a function  $c(t) \in \mathcal{L}$ ,  $c(t) \neq 0$ , such that

$$(3.2) \quad e^{-(at+b)p} (c(t) \times F(p, t)) \underset{p \rightarrow \infty}{\Rightarrow} 0 .$$

(3.3) Theorem. A function  $F(p, t)$  is feasible iff :

1º  $F(p, t) \in S$ ,

2º there exist real numbers  $a, b$ ,  $0 \leq a < t$ ,  $b \geq 0$  and a function  $c(t) \in \mathcal{L}$ ,  $c(t) \neq 0$ , such that

$$(3.4) \quad \underset{0}{\overset{t}{\int}} e^{-(at+b)p} c(t-u) F(p, u) du \underset{p \rightarrow \infty}{\Rightarrow} 0 .$$

Proof. If a function  $F(p, t)$  is feasible then, by Definition (3.1), we have  $F(p, t) \in S$  and (3.2). It follows that

$$\frac{1}{p} e^{-(at+b)p} (c(t) \times F(p, t)) \underset{p \rightarrow \infty}{\Rightarrow} 0 ,$$

which is equivalent to (3.4).

Conversely, if we have  $F(p, t) \in S$  and (3.4) then for any  $q > b$  we have

$$e^{-(at+q)p} \rho \int_0^t c(t-u) F(p, u) du \Rightarrow 0 , \quad p \rightarrow \infty$$

that is,

$$e^{-(at+q)p} (c(t) \times F(p, t)) \Rightarrow 0 \quad p \rightarrow \infty$$

and, by Definition (3.1), the function  $F(p, t)$  is feasible. •

(3.5) Theorem. For any functions  $F_1(p, t), F_2(p, t) \in S$  and any real numbers  $\alpha, \beta, \gamma, \delta$  we have

$$\begin{aligned} (3.6) \quad & e^{(\alpha t + \beta)p} F_1(p, t) \times e^{(\gamma t + \delta)p} F_2(p, t) = \\ & = e^{(\gamma t + \beta + \delta)p} (e^{(\alpha - \gamma)p} F_1(p, t) \times F_2(p, t)) = \\ & = e^{(\alpha t + \beta + \delta)p} (F_1(p, t) \times e^{(\gamma - \alpha)p} F_2(p, t)) \end{aligned}$$

and

$$\begin{aligned} (3.7) \quad & e^{(\alpha t + 2\beta)p} (F_1(p, t) \times F_2(p, t)) = \\ & = e^{(\alpha t + \beta)p} F_1(p, t) \times e^{(\alpha t + \beta)p} F_2(p, t) . \end{aligned}$$

Proof. We have

$$\begin{aligned} & e^{(\alpha t + \beta)p} F_1(p, t) \times e^{(\gamma t + \delta)p} F_2(p, t) = \\ & = \rho \int_0^t e^{(\alpha t - \alpha u + \beta)p} F_1(p, t-u) e^{(\gamma u + \delta)p} F_2(p, u) du = \\ & = e^{(\gamma t + \beta + \delta)p} \rho \int_0^t e^{(\alpha - \gamma)(t-u)p} F_1(p, t-u) F_2(p, u) du = \\ & = e^{(\gamma t + \beta + \delta)p} (e^{(\alpha - \gamma)p} F_1(p, t) \times F_2(p, t)) . \end{aligned}$$

We prove the second part of the formula (3.6) in the same way. The formula (3.7) follows from (3.6) when we put  $\alpha=\gamma$  and  $\beta=\delta$ . •

(3.8) Theorem. If we replace the numbers  $a, b$  in a true formula (3.2) or (3.4) by any numbers  $r > a, s > b$ , where  $r \neq s$ , then we obtain a true formula.

Proof. Introducing  $u := r-a > 0$  and  $v := s-b$  we obtain from (3.2)

$$e^{-(rt+st)p} (c(t) \times F(p, t)) = e^{-(ut+vt)p} e^{-(at+bt)p} (c(t) \times F(p, t)) \xrightarrow[p \rightarrow \infty]{} 0 .$$

The proof for (3.4) is analogous. •

(3.9) Theorem. If for any function  $f(t) \in \mathcal{L}$ ,  $f(t) \neq 0$ , we replace the function  $c(t) \times f(t)$  in a true formula (3.2) or (3.4) by  $1/p c(t) \times f(t) \times F(p, t)$  then we obtain a true formula.

Proof. If (3.2) is true then, by (3.6) and Theorem (2.26), also

$$\begin{aligned} e^{-(at+bt)p} (1/p c(t) \times f(t) \times F(p, t)) &= \\ &= 1/p e^{-atp} f(t) \times e^{-(at+bt)p} (c(t) \times F(p, t)) \xrightarrow[p \rightarrow \infty]{} 0 . \end{aligned}$$

The formula (3.4) is equivalent to

$$1/p e^{-(at+bt)p} (c(t) \times F(p, t)) \xrightarrow[p \rightarrow \infty]{} 0$$

and the proof is analogous. •

(3.10) Definition. The set  $\mathcal{F}$  is the set of all feasible functions. •

(3.11) Theorem. The set  $F$  forms a **commutative ring**, that is, a subring of the ring  $S$ .

Proof. If  $F_1(p, t), F_2(p, t) \in F$  then, according to Definition (3.1) and Theorem (2.23),  $F_1(p, t), F_2(p, t), F_1(p, t) \pm F_2(p, t), F_1(p, t) \times F_2(p, t) \in S$  and there exist real numbers  $a_1, a_2, b_1, b_2$ ,  $0 \leq a_j < 1$ ,  $0 \leq a_2 < 1$ ,  $b_1 \geq 0$ ,  $b_2 \geq 0$  and functions  $c_1(t), c_2(t) \in C$ ,  $c_1(t), c_2(t) \neq 0$ , such that

$$(3.12) \quad e^{-(a_j t + b_j)p} (c_j(t) \times F_j(p, t)) \underset{p \rightarrow \infty}{\Rightarrow} 0 \quad \text{for } j = 1, 2.$$

By Theorems (3.8) and (3.9), we have for  $a := \max(a_1, a_2)$ ,  $b := \max(b_1, b_2)$ ,

$$e^{-(at + b)p} \left( \int_0^t c_1(t-u) c_2(u) du \times F_j(p, t) \right) \underset{p \rightarrow \infty}{\Rightarrow} 0 \quad \text{for } j = 1, 2$$

and hence

$$e^{-(at + b)p} \left( \int_0^t c_1(t-u) c_2(u) du \times (F_1(p, t) \pm F_2(p, t)) \right) \underset{p \rightarrow \infty}{\Rightarrow} 0.$$

It follows that the functions  $F_1(p, t) + F_2(p, t)$  and  $F_1(p, t) - F_2(p, t)$  are feasible.

We have furthermore, in view of (3.12), Theorem (2.31) and the formula (3.7),

$$\begin{aligned} & e^{-(at + 2b)p} \left( \int_0^t c_1(t-u) c_2(u) du \times (F_1(p, t) \times F_2(p, t)) \right) = \\ &= 1/p e^{-(at + 2b)p} (c_1(t) \times c_2(t) \times F_1(p, t) \times F_2(p, t)) = \\ &= 1/p \left\{ e^{-(at + b)p} (c_1(t) \times F_1(p, t)) \times e^{(at + b)p} (c_2(t) \times F_2(p, t)) \right\} \underset{p \rightarrow \infty}{\Rightarrow} 0. \end{aligned}$$

It follows that the function  $F_1(p, t) \times F_2(p, t)$  is feasible.

This completes the proof. •

(3.13) Theorem. We have  $r(t) \in F$  iff  $r(t) \in S$ , that is, iff  $r(t)$  is defined for almost all  $t \geq 0$  and integrable in the Lebesgue sense on every finite interval  $0 \leq t \leq T$ .

Proof. If  $f(t) \in F$  then, by definition,  $f(t) \in S$ . Conversely, if  $f(t) \in S$  then for  $a = 0$ , any  $b > 0$  and  $c(t) := 1$  we have

$$e^{-(at+b)p} \int_0^t c(t-u) f(u) du = e^{-bp} \int_0^t f(u) du \xrightarrow[p \rightarrow \infty]{} 0.$$

By Theorem (3.3),  $f(t) \in F$ . •

(3.14) Theorem. We have  $g(p) \in F$  iff there exists a real non-negative number  $k$  such that

$$(3.15) \quad e^{-kp} g(p) \xrightarrow[p \rightarrow \infty]{} 0.$$

Proof. If  $g(p) \in F$  then, by Theorem (3.3), there exist real numbers  $a, b$ ,  $0 \leq a < 1$ ,  $b \neq 0$  and a function  $c(t) \in L$ ,  $c(t) \neq 0$ , such that

$$(3.16) \quad e^{-(at+b)p} \int_0^t c(t-u) g(p) du \xrightarrow[p \rightarrow \infty]{} 0,$$

that is,

$$(3.17) \quad e^{-(at+b)p} \int_0^t c(u) du \xrightarrow[p \rightarrow \infty]{} 0.$$

Let  $t_0$  be a real number such that the integral in (3.17) is not zero for  $t = t_0$ . Then for  $k := at_0 + b$  we obtain (3.15).

Conversely, if we have (3.15) then for any function  $c(t) \in L$ ,  $c(t) \neq 0$ , and  $a = 0$ ,  $b = k$  we have (3.17), that is (3.16), and by virtue of Theorems (3.3) and (2.28),  $g(p) \in F$ . •

(3.18) Theorem. For any real  $r$  we have  $p^r \in F$ .

Proof. The theorem follows from Theorem (3.14), because for any  $k > 0$  we have

$$e^{-kp} p^r \xrightarrow[p \rightarrow \infty]{} 0. \quad \bullet$$

(3.19) Theorem. For any real  $r$  we have  $e^{rp} \in F$ .

Proof. The theorem follows from Theorem (3.14), because for any  $k > |r|$  we have

$$e^{-kp} e^{rp} = e^{-(k-r)p} \xrightarrow[p \rightarrow \infty]{} 0 . \quad \bullet$$

(3.20) Theorem. If  $F(p, t) \in S$  then for any function  $f(t) \in L$ ,  $f(t) \neq 0$ , we have

$$(3.21) \quad f(t) \times F(p, t) \in F \quad \text{iff} \quad F(p, t) \in F .$$

Proof. If  $F(p, t) \in F$  then, by Theorems (3.13) and (3.11),  $f(t) \times F(p, t) \in F$ . Conversely, if  $f(t) \times F(p, t) \in F$  then, by Definition (3.1), there exist real numbers  $a, b$ ,  $0 \leq a < 1$ ,  $b \geq 0$  and a function  $c(t) \in L$ ,  $c(t) \neq 0$ , such that

$$e^{-(at+b)p} (c(t) \times f(t) \times F(p, t)) \xrightarrow[p \rightarrow \infty]{} 0 .$$

Hence

$$\begin{aligned} 1/p e^{-(at+b)p} (c(t) \times f(t) \times F(p, t)) &= e^{-(at+b)p} (1/p c(t) \times f(t) \times F(p, t)) = \\ &= e^{-(at+b)p} \left( \int_0^t c(t-u) f(u) du \times F(p, t) \right) \xrightarrow[p \rightarrow \infty]{} 0 , \end{aligned}$$

which means that  $F(p, t) \in F$ .  $\bullet$

(3.22) Theorem. We have, for any functions  $g(p)$ ,  $F(p, t) \in S$ ,

$$g(p) \times F(p, t) \in F \quad \text{iff} \quad g(p) F(p, t) \in F .$$

Proof. By virtue of Theorems (2.23) and (2.29), we have  $g(p) \times F(p, t) \in S$  and  $g(p) F(p, t) \in S$ . Since

$$g(p) \times F(p, t) = I \times g(p) F(p, t) ,$$

our assertion follows from Theorem (3.20). •

(3.23) Theorem. For any real  $r$  we have

$$F(p, t) \in F \quad \text{iff} \quad p^r F(p, t) \in F .$$

Proof. By Theorems (3.18) and (3.11), if  $F(p, t) \in F$  then  $p^r \times F(p, t) \in F$  and, in view of Theorem (3.22),  $p^r F(p, t) \in F$ .

Conversely, if  $p^r F(p, t) \in F$  then, by virtue of the first part of our theorem,  $F(p, t) = p^{-r} p^r F(p, t) \in F$ . •

(3.24) Theorem. We have

$$(3.25) \quad F(p, t) \in F \quad \text{iff} \quad p \int_0^t F(p, u) du \in F .$$

Proof. In view of Theorem (2.31), our theorem follows from the fact that

$$p \int_0^t F(p, u) du = I \times F(p, t) ,$$

according to Theorem (3.20) . •

(3.26) Theorem. We have

$$(3.27) \quad F(p, t) \in F \quad \text{iff} \quad \int_0^t F(p, u) du \in F .$$

Proof. In view of Theorem (2.31), our theorem follows from Theorems (3.24) and (3.23). •

## IV. Distributions and congruent functions

(4.1) Definition. The class  $Z$  is the set of all functions  $Z(p, t) \in S$  satisfying the condition

$$(4.2) \quad e^{pt} \times Z(p, t) \in F .$$

(4.3) Theorem.  $Z$  is a subset of  $F$ .

Proof. For any  $Z(p, t) \in Z$  define the function

$$(4.4) \quad F(p, t) := Z(p, t) + e^{pt} \times Z(p, t) .$$

Since

$$\begin{aligned} F(p, t) - 1 \times F(p, t) &= Z(p, t) + e^{pt} \times Z(p, t) - 1 \times Z(p, t) - \\ &\quad - 1 \times e^{pt} \times Z(p, t) = \\ &= Z(p, t) + e^{pt} \times Z(p, t) - 1 \times Z(p, t) - \\ &\quad - (e^{pt} - 1) \times Z(p, t) = Z(p, t) , \end{aligned}$$

we have

$$(4.5) \quad Z(p, t) = F(p, t) - 1 \times F(p, t) .$$

It follows from (4.4) and (4.5) that

$$(4.6) \quad e^{pt} \times Z(p, t) = 1 \times F(p, t) \in F .$$

By virtue of Theorem (3.20), we have  $F(p, t) \in F$  and, in view of (4.5),  $Z(p, t) \in F$ . This completes the proof. •

(4.7) Theorem. The class  $Z$  forms an ideal of the ring  $F$ .

Proof. First, we shall show that  $Z$  forms a group with respect to addition. Let  $Z_1(p, t), Z_2(p, t) \in Z$ . Thus

$$e^{pt} \times Z_1(p, t) \in F \quad \text{and} \quad e^{pt} \times Z_2(p, t) \in F.$$

Hence

$$e^{pt} \times (Z_1(p, t) + Z_2(p, t)) = e^{pt} \times Z_1(p, t) + e^{pt} \times Z_2(p, t) \in F,$$

which means that  $Z_1(p, t) + Z_2(p, t) \in Z$ . Moreover, it follows from (4.2) that  $Z(p, t) \in Z$  iff  $-Z(p, t) \in Z$ . Thus  $Z$  forms a group with respect to addition.

Now, it is sufficient to show that if  $Z(p, t) \in Z$  then, for every  $F(p, t) \in F$ , we have  $Z(p, t) \times F(p, t) \in Z$ . Indeed, (4.2) implies

$$e^{pt} \times Z(p, t) \times F(p, t) \in F,$$

which means that  $Z(p, t) \times F(p, t) \in Z$ . •

(4.8) Definition. The distribution ring \*) or, simply, the ring  $\mathcal{B}$  is the residue class ring  $F/Z$  into which the ideal  $Z$  divides the ring  $F$ . •

(4.9) Definition. A distribution is any residue class belonging to the ring  $\mathcal{B}$ . •

\*) Here, the word "distribution" has not the meaning used in the theory of distributions.

(4.10) Definition. Each function  $F(p, t) \in F$  belonging to a distribution  $A$  is to be called a **representative** of this distribution and is written in the form

$$A = \{ F(p, t) \} . \quad \bullet$$

(4.11) Definition. Any functions  $F_1(p, t), F_2(p, t) \in F$  are said to be **congruent** iff they are representatives of the same distribution and in such a case we write

$$(4.12) \quad F_1(p, t) \equiv F_2(p, t) . \quad \bullet$$

It follows that (4.12) holds iff  $F_1(p, t) - F_2(p, t) \in Z$ . The congruence  $F(p, t) = 0$  means that  $F(p, t) \in Z$ .

(4.13) Theorem. The relation (4.12) is a congruence modulo  $Z$  and, therefore, it has the following properties :

$$(4.14) \quad F(p, t) \equiv F(p, t) ,$$

$$(4.15) \quad \text{if } F_1(p, t) \equiv F_2(p, t) \text{ then } F_2(p, t) \equiv F_1(p, t) ,$$

$$(4.16) \quad \text{if } F_1(p, t) \equiv F_2(p, t) \text{ and } F_2(p, t) \equiv F_3(p, t) \text{ then } F_1(p, t) \equiv F_3(p, t) ,$$

$$(4.17) \quad \text{if } F_j(p, t) \equiv G_j(p, t) \text{ for } j=1, \dots, n \text{ then } \sum_{j=1}^n F_j(p, t) \equiv \sum_{j=1}^n G_j(p, t) ,$$

$$(4.18) \quad \text{if } F_j(p, t) \equiv G_j(p, t) \text{ for } j=1, \dots, n \text{ then}$$

$$F_1(p, t) \times \dots \times F_n(p, t) = G_1(p, t) \times \dots \times G_n(p, t) ,$$

(4.19) if  $F_j(p, t) = \theta_j(p, t)$  and  $H_j(p, t) \in F$  for  $j=1, \dots, n$  then

$$\sum_{j=1}^n (H_j(p, t) \times F_j(p, t)) = \sum_{j=1}^n (H_j(p, t) \times \theta_j(p, t)).$$

Proof. All the above properties are well-known properties of any residue class ring. •

(4.20) Theorem. For every  $F(p, t) \in F$  we have

$$(4.21) \quad F(p, t) = p \int_0^t F(p, u) du.$$

Proof. In view of Theorem (3.24), it is sufficient to prove that

$$Z(p, t) := F(p, t) - p \int_0^t F(p, u) du = F(p, t) - 1 \times F(p, t) = Z.$$

We have, according to Definition (4.1),

$$\begin{aligned} e^{pt} \times Z(p, t) &= e^{pt} \times F(p, t) - e^{pt} \times 1 \times F(p, t) = \\ &= e^{pt} \times F(p, t) - (e^{pt} - 1) \times F(p, t) = \\ &= 1 \times F(p, t) \in F. \end{aligned}$$

This completes the proof. •

(4.22) Theorem. A function  $Z(p, t) \in S$  belongs to ideal  $Z$  iff there exists a function  $F(p, t) \in F$  such that

$$(4.23) \quad Z(p, t) = F(p, t) - 1 \times F(p, t) = F(p, t) - p \int_0^t F(p, u) du.$$

Proof. By Theorem (4.3), if  $Z(p, t) \in Z$  then the function (4.4) satisfies (4.5). Conversely, by Theorem (4.20), the function (4.23) belongs to  $Z$ . •

Theorem (4.22) shows that  $Z$  is the smallest subset of  $F$  ensuring the congruence (4.21) for every function  $F(p, t) \in F$ .

(4.24) Theorem. If  $g(p) \in F$  and  $F(p, t) \in F$  then

$$(4.25) \quad g(p) \times F(p, t) \equiv g(p) F(p, t) .$$

Proof. By virtue of Theorem (3.22)  $g(p) F(p, t) \in F$  and by Theorem (4.20)

$$g(p) \times F(p, t) = p \int_0^t g(p) F(p, u) du \equiv g(p) F(p, t) ,$$

which was to be proved. •

(4.26) Theorem. If  $F_1(p, t) \equiv F_2(p, t)$  and  $g(p) \in F$  then

$$(4.27) \quad g(p) F_1(p, t) \equiv g(p) F_2(p, t) .$$

Proof. According to (4.19), we have  $g(p) \times F_1(p, t) \equiv g(p) \times F_2(p, t)$ . Hence, by Theorem (4.24), we obtain (4.27). •

(4.28) Theorem. We have  $F(p, t) = 0$  iff

$$(4.29) \quad p \int_0^t F(p, u) du \equiv 0 .$$

Proof. The theorem follows from (4.21). •

(4.30) Theorem. We have  $F(p, t) = 0$  iff

$$(4.31) \quad \int_0^t F(p, u) du = 0 .$$

Proof. By Theorems (4.26) and (3.18), multiplying (4.29) on both sides by  $t/p$ , we obtain (4.31) and, multiplying (4.31) on both sides by  $p$ , we obtain (4.29). •

(4.32) Theorem. For every function  $f(t) \in F$  we have

$$(4.33) \quad f(t) = 0$$

iff

$$(4.34) \quad f(t) = 0 \text{ a.e.}$$

Proof. It is sufficient to prove the theorem only in the case when  $f(t) \in \mathcal{L}$  and condition (4.34) may be replaced by

$$(4.35) \quad f(t) = 0 ,$$

because, having it, we have by Theorem (4.30), for any function  $f(t) \in F$ ,

$$f(t) = 0 \text{ iff } \int_0^t f(u) du = 0 \text{ iff } \int_0^t f(u) du = 0 \text{ iff } f(t) = 0 \text{ a.e.}$$

Moreover, it is sufficient to prove that (4.33) implies (4.35), because it is evident that (4.35) implies

$$e^{pt} \times f(t) = 0 \in F ,$$

which is equivalent to (4.33). Thus it is sufficient to prove that

$$(4.36) \quad e^{pt} \times f(t) \in F$$

implies (4.35). Let us suppose that

$$(4.37) \quad f(t) \neq 0 .$$

Then, by virtue of Theorem (3.20), we have (4.36) iff

$$e^{pt} \in F ,$$

that is, there exist real numbers  $a, b$ ,  $0 \leq a < 1$ ,  $b \geq 0$  and a function  $c(t) \in \mathcal{L}$ ,  $c(t) \neq 0$ , such that, according to Theorem (3.3),

$$(4.38) \quad e^{-(at+b)p} \int_0^t c(t-u) e^{pu} du \Rightarrow 0 \quad p \rightarrow \infty$$

This implies

$$\begin{aligned} & \lim_{p \rightarrow \infty} \int_0^t e^{p(u-at-b)} c(t-u) du = \lim_{p \rightarrow \infty} \int_{-at-b}^{t-at-b} e^{pu} c(t-u) du \\ &= \lim_{p \rightarrow \infty} \int_0^{t-at-b} e^{pu} c(t-at-b-u) du + \lim_{p \rightarrow \infty} \int_{-at-b}^0 e^{pu} c(t-at-b-u) du = 0. \end{aligned}$$

Now, let  $t > b/(1-a)$ . Since  $c(t) \in \mathcal{L}$ , the function  $c(t-at-b-u)$  is bounded in the interval  $-at-b \leq u \leq 0$  for every  $b/(1-a) \leq t \leq T$ . Therefore

$$\lim_{p \rightarrow \infty} \int_{-at-b}^0 e^{pu} c(t-at-b-u) du = 0.$$

Thus, we obtain for every  $t > b/(1-a)$

$$\lim_{p \rightarrow \infty} \int_0^{t-at-b} e^{pu} c(t-at-b-u) du = 0,$$

that is, for every  $t > 0$ ,

$$\lim_{p \rightarrow \infty} \int_0^t e^{pu} c(t-u) du = 0.$$

By virtue of the Theorem on Bounded Moments (see [3], p.395, or [5], p.18), we obtain  $c(t-u) = 0$  for every  $u$  from the interval  $0 \leq u \leq t$ ; that is,  $c(u)$  equals zero for every  $u$  from that interval. Since  $t$  may be arbitrarily great, we obtain  $c(t) = 0$ , which contradicts the assumption. Therefore (4.37) is not possible and we obtain (4.35). This completes the proof. •

(4.39) Theorem. The congruence

$$f_1(t) \equiv f_2(t)$$

holds iff

$$f_1(t) \stackrel{\text{a.s.}}{=} f_2(t).$$

Proof. We obtain this immediately, when we replace the function  $f(t)$  in Theorem (4.32) by  $f_1(t) - f_2(t)$ . •

(4.40) Theorem. For any  $f(t) \in \mathcal{C}$ ,  $f(t) \neq 0$ , we have

$$(4.41) \quad f(t) \times F(p, t) \equiv 0 \quad \text{iff} \quad F(p, t) \equiv 0 .$$

Proof. By definition,  $f(t) \times F(p, t) \equiv 0$  means that

$$e^{pt} \times f(t) \times F(p, t) \in F .$$

By Theorem (3.20), this is equivalent to

$$e^{pt} \times F(p, t) \in F ,$$

which means that  $F(p, t) \equiv 0$ . •

(4.42) Theorem. If  $F(p, t)$ ,  $G(p, t) \in F$  satisfy the congruence

$$(4.43) \quad F(p, t) \times G(p, t) \equiv 0$$

and there exist functions  $H(p, t) \in F$ ,  $f(t) \in \mathcal{C}$ ,  $f(t) \neq 0$ , such that

$$(4.44) \quad G(p, t) \times H(p, t) \equiv f(t) ,$$

then

$$F(p, t) \equiv 0 .$$

Proof. Multiplying (4.43) on both sides by  $H(p, t)$ , we obtain by (4.41)  $F(p, t) \times f(t) \equiv 0$  and then, by Theorem (4.40),  $F(p, t) \equiv 0$ , which completes the proof. •

(4.45) Theorem. For any  $f(t) \in \mathcal{E}$ ,  $f(t) \neq 0$ , we have

$$(4.46) \quad f(t) \times F_1(p, t) = f(t) \times F_2(p, t) \quad \text{iff} \quad F_1(p, t) = F_2(p, t).$$

Proof. We obtain this, when replacing the function  $F(p, t)$  in Theorem (4.40) by  $F_1(p, t) - F_2(p, t)$ . •

(4.47) Theorem. We have

$$(4.48) \quad g(p) \equiv 0 \quad \text{and} \quad g(p) \in F \quad \text{iff} \quad e^{pt} g(p) \in F.$$

Proof. The theorem follows from Definition (4.1) and Theorem (4.24). •

(4.49) Theorem. If  $g(p) \in F$  and

$$(4.50) \quad g(p) \equiv 0$$

then for every  $t \geq 0$ ,

$$(4.51) \quad \liminf_{p \rightarrow \infty} e^{pt} |g(p)| = 0.$$

Proof. According to Theorem (4.47), the congruence (4.50) means that

$$e^{pt} g(p) \in F.$$

By Theorem (3.3), there exist real numbers  $a, b$ ,  $0 \leq a < 1$ ,  $b \geq 0$  and a function  $c(t) \in \mathcal{E}$ ,  $c(t) \neq 0$ , such that

$$(4.52) \quad \left| g(p) \right| \int_0^t c(t-u) e^{p(u-a)t-b} du \Rightarrow 0.$$

Now, let us suppose that (4.51) were false for  $t = s > 0$ . Then, there would exist two real positive numbers  $r$  and  $s$  such that

$$\left| e^{ps} g(p) \right| > r \quad \text{for any natural number } p > s.$$

Then it would follow from (4.52) that

$$e^{-(at+b+sp)} \int_0^t c(t-u) e^{pu} du \xrightarrow[p \rightarrow \infty]{} 0$$

and, analogously to (4.38), we would obtain  $c(t) = 0$ , against the assumption. It follows that (4.51) must be true. •

(4.53) Theorem. If for every  $t \geq 0$

$$(4.54) \quad \lim_{p \rightarrow \infty} e^{pt} g(p) = 0$$

then  $g(p) \in F$  and

$$(4.55) \quad g(p) = 0 .$$

Proof. By assumption, we have for  $t = 0$

$$\lim_{p \rightarrow \infty} g(p) = 0$$

and, by Theorem (3.14),  $g(p) \in F$ .

We have, further, for any real  $k > 0$ ,  $T > 0$  and every  $t$  from the interval  $0 \leq t \leq T$ ,

$$\begin{aligned} |e^{-kp} (1 \times e^{pt} g(p))| &= |e^{-kp} g(p)| \left| \int_0^t p \int e^{pu} du \right| \leq \\ &\leq |e^{-kp} g(p)| \left| \int_0^T p \int e^{pu} du \right| = |e^{-kp} g(p) (e^{pT} - 1)| \leq \\ &\leq |e^{-kp}| |e^{pT} g(p)| + |e^{-kp} g(p)| \xrightarrow[p \rightarrow \infty]{} 0 . \end{aligned}$$

It follows that

$$e^{-kp} (1 \times e^{pt} g(p)) \xrightarrow[p \rightarrow \infty]{} 0$$

and, by Definition (3.1),  $e^{pt} g(p) \in F$ . By virtue of Theorem (4.47), we obtain (4.55). •

(4.56) Example. For any real  $r > t$  we have

$$e^{-\langle p^r \rangle} = 0$$

because for every  $t \geq 0$  we have

$$\lim_{p \rightarrow \infty} e^{pt} e^{-\langle p^r \rangle} = 0 .$$

(4.57) Theorem. If  $F(p, t) \in F$  and for any real  $r$

$$(4.58) \quad F(p, t) = 0 \quad \begin{cases} \text{for } 0 \leq t \leq r \text{ when } r > 0 , \\ \text{for } r \leq t \leq 0 \text{ when } r < 0 , \end{cases}$$

then

$$(4.59) \quad S(p, t) := F(p, t+r) \in F$$

and

$$(4.60) \quad S(p, t) = 0 \quad \begin{cases} \text{for } -r \leq t \leq 0 \text{ when } r > 0 , \\ \text{for } 0 \leq t \leq -r \text{ when } r < 0 . \end{cases}$$

Proof. If  $r = 0$  then (4.59) is evident. Therefore, let us assume that  $r \neq 0$ . Since (4.60) follows immediately from (4.58), it is sufficient to prove (4.59).

By assumption,  $F(p, t) \in S$ . It follows that  $S(p, t) \in S$  too. Moreover, by assumption, there exist real numbers  $a, b$ ,  $0 \leq a < 1$ ,  $b \neq 0$ , and a function  $c(t) \in C$ ,  $c(t) \neq 0$ , such that, according to Theorem (3.3),

$$e^{-\langle at + b \rangle p} \int_0^t c(t-u) F(p, u) du \Rightarrow 0 .$$

In view of (4.58), we obtain

$$e^{-\langle at + ar + b \rangle p} \int_r^{t+r} c(t+r-u) F(p, u) du = e^{-\langle at + ar + b \rangle p} \int_0^t c(t-u) S(p, u) du \Rightarrow 0 .$$

which means that  $S(p, t) \in F$ . •

(4.61) Theorem. If, for any real  $r$ ,  $F(p, t) \in F$  is a function satisfying the condition (4.58) and  $\delta(p, t)$  is defined by (4.59) then

$$(4.62) \quad F(p, t) = \delta(p, t) e^{-tp} .$$

Proof. It follows from Theorem (3.3) that there exist real numbers  $a, b$ ,  $0 \leq a < 1$ ,  $b \geq 0$  and a function  $c(t) \in \mathcal{C}$ ,  $c(t) \neq 0$ , such that

$$e^{-(at+b)p} \int_0^t c(t-u) F(p, u) du \Rightarrow 0 \quad p \rightarrow \infty .$$

Hence

$$(4.63) \quad \left| \int_0^r e^{-(at+b)p} \int_0^{t-u} c(t-v-u) F(p, v) dv | du \right| \Rightarrow 0 \quad p \rightarrow \infty .$$

But, in view of (4.58),

$$\begin{aligned} & \left| \int_0^r e^{-(at+b)p} \int_0^{t-u} c(t-v-u) F(p, v) dv | du \right| = \\ &= \left| \int_0^r e^{-(at+b)p} \int_{-v}^{t-u} c(t-v-u) F(p, v) dv | du \right| \geq \\ &\geq \left| \int_0^r e^{p(v+|r|)} \left| \int_{-v}^{t-u} e^{-(at+b)p} c(t-v-u) F(p, v) dv \right| du \right| \stackrel{w=u-v}{=} \\ &= \left| \int_0^r e^{p(v+at+b+|r|)} \int_0^t c(t-u) F(p, u-v) du | du \right| \geq \\ &\geq \left| \int_0^r \int_0^t e^{p(v+at+b+|r|)} c(t-u) F(p, u-v) du | du \right| = \\ &= \left| e^{-(at+b+|r|)p} \int_0^t \int_0^r e^{pv} F(p, u-v) du | du \right| \stackrel{w=u-v}{=} \\ &= \left| e^{-(at+b+|r|)p} \int_0^t c(t-u) \int_{u+r}^u e^{p(u-w)} F(p, w) dw | du \right| \end{aligned}$$

and, in view of (4.63),

$$e^{-(at+b+|r|)p} \int_0^t c(t-u) \int_{u+r}^u e^{p(u-w)} F(p, w) dw | du \Rightarrow 0 \quad p \rightarrow \infty .$$

By virtue of Theorems (3.3) and (3.23), we have

$$(4.64) \quad p \int_{t+r}^t e^{p(t-u)} F(p, u) du \in F .$$

Hence, in view of (4.59),

$$p \int_0^t e^{p(t-u)} F(p, u) du - p \int_0^{t+r} e^{p(t-u)} F(p, u) du \in F ,$$

$$p \int_0^t e^{p(t-u)} F(p, u) du - p \int_{-r}^t e^{p(t-u-r)} F(p, u+r) du \in F ,$$

$$p \int_0^t e^{p(t-u)} F(p, u) du - p \int_0^t e^{p(t-u)} e^{-tp} G(p, u) du \in F ,$$

$$e^{pt} \times (F(p, t) - e^{-tp} G(p, t)) \in F .$$

It means that  $F(p, t) - e^{-tp} G(p, t) \equiv 0$ , which is equivalent to (4.62). •

(4.65) Theorem. If

$$F(p, t) \equiv g(p) ,$$

where  $F(p, t)$ ,  $g(p) \in F$ , and we put  $F(p, t) = 0$  for  $-r \leq t < 0$ ,  $r$  being any real positive number, then

$$(4.66) \quad F(p, t-r) = g(p) e^{-rp} .$$

Proof. By virtue of Theorems (4.57) and (4.61), we have

$$g(p) \equiv F(p, t) \equiv G(p, t) e^{tp} = F(p, t-r) e^{rp} .$$

Hence, by Theorems (3.19) and (4.26), we obtain (4.66). •

(4.67) Theorem. If

$$F(p, t) = g(p)$$

where  $F(p, t), g(p) \in F$  and

$$(4.68) \quad F(p, t) = 0 \quad \text{for} \quad 0 \leq t < r,$$

$r$  being any real positive number, then

$$(4.69) \quad F(p, t+r) = g(p) e^{rp}.$$

Proof. We prove this theorem analogously to the previous one. •

(4.70) Theorem. If  $F_1(p, t), F_2(p, t), g_1(p), g_2(p) \in F$  and

$$(4.71) \quad F_1(p, t) = g_1(p), \quad F_2(p, t) = g_2(p),$$

then

$$(4.72) \quad F_1(p, t) \times F_2(p, t) = g_1(p) g_2(p).$$

Proof. By virtue of the property (4.18) of congruences we have

$$(4.73) \quad F_1(p, t) \times F_2(p, t) = g_1(p) \times g_2(p)$$

and, by Theorem (4.24),

$$(4.74) \quad g_1(p) \times g_2(p) = g_1(p) g_2(p).$$

The congruences (4.73) and (4.74) imply (4.72). •

(4.75) Theorem. If  $p_0$  is a real non-negative number such that for every natural  $p > p_0$  a function  $F(p, t)$  is absolutely continuous in every finite interval  $0 \leq t \leq T$ , then  $F(p, t), \partial/\partial t F(p, t) \in F$  and

$$(4.76) \quad \frac{d}{dt} F(p, t) = p F(p, t) - p F(p, 0),$$

where also  $F(p, 0) \in F$ .

Proof. By virtue of Theorem (3.13), we have  $F(p, t) \in F$ . As we know from the theory of the Lebesgue integral,  $F(p, t)$  has a derivative  $\frac{d}{dt} F(p, t)$  for almost every  $t > 0$ , when  $p > p_0$  is an integer, and this derivative is integrable in the Lebesgue sense on every finite interval  $0 \leq t \leq T$ . It follows that  $\frac{d}{dt} F(p, t) \in F$ . Moreover, as we know from the theory of the Lebesgue integral,

$$(4.77) \quad \int_0^t \frac{d}{du} F(p, u) du = F(p, t) - F(p, 0).$$

It follows from Theorem (3.26), that

$$\int_0^t \frac{d}{du} F(p, u) du \in F.$$

Thus (4.77) implies  $F(p, 0) \in F$  and

$$p \int_0^t \frac{d}{du} F(p, u) du = p F(p, t) - p F(p, 0).$$

Hence, by Theorem (4.20), we obtain (4.76). •

(4.78) Theorem. If  $p_0$  is a real non-negative number such that for every natural  $p > p_0$  functions  $F(p, t)$ ,  $\frac{d}{dt} F(p, t)$ , ...,  $\frac{d^{k-1}}{dt^{k-1}} F(p, t)$  are absolutely continuous in every finite interval  $0 \leq t \leq T$  then  $F(p, t)$ ,  $\frac{d}{dt} F(p, t)$ , ...,  $\frac{d^k}{dt^k} F(p, t) \in F$  and

$$(4.79) \quad \frac{d^k}{dt^k} F(p, t) = p^k F(p, t) - p^k F(p, 0) - p^{k-1} F'(1)(p, 0) - \dots - p F^{(k-1)}(p, 0),$$

where

$$(4.80) \quad F^{(j)}(p, 0) := [\frac{d^j}{dt^j} F(p, t)]_{t=0} \in F \quad \text{for } j = 1, \dots, k-1.$$

Proof. We obtain (4.79) when applying Theorem (4.75)  $k$  times. •

(4.81) Theorem. We have

$$(4.82) \quad g(p) := p \int_0^\infty e^{-pu} F(p, u) du \in F$$

and

$$(4.83) \quad g(p) = F(p, t),$$

where  $F(p, t) \in F$ , iff

$$(4.84) \quad P(p, t) := p \int_t^\infty e^{p(t-u)} F(p, u) du \in F.$$

Proof. Let

$$(4.85) \quad Z(p, t) := P(p, t) - p \int_0^t F(p, u) du.$$

Then

$$\begin{aligned} Z(p, t) &= p \int_t^\infty e^{p(t-u)} F(p, u) du - p^2 \iint_0^t e^{p(u-v)} F(p, v) du dv = \\ &= p \int_0^\infty e^{p(t-u)} F(p, u) du - p \int_0^t e^{p(t-u)} F(p, u) du - \\ &\quad - p^2 \iint_0^t e^{p(u-v)} F(p, v) du dv + p^2 \iint_0^t e^{p(u-v)} F(p, v) du dv = \\ &= e^{pt} g(p) - p \int_0^t e^{p(t-u)} F(p, u) du - p g(p) \int_0^t e^{pu} du + \\ &\quad + p \int_0^t e^{-pu} F(p, u) \int_u^t p e^{pv} du dv = \\ &= e^{pt} g(p) - p \int_0^t e^{p(t-u)} F(p, u) du - e^{pt} g(p) + g(p) + \\ &\quad + p \int_0^t e^{p(t-u)} F(p, u) du - p \int_0^t F(p, u) du = \\ &= g(p) - p \int_0^t F(p, u) du. \end{aligned}$$

By virtue of Theorem (3.24), we have

$$p \int_0^t F(p, u) du \in F$$

and also

$$(4.86) \quad P(p, t) \in F \quad \text{iff} \quad p \int_0^t P(p, u) du \in F .$$

First, let us suppose that (4.82) and (4.83) are true. Then, by Theorem (4.20), we have

$$Z(p, t) = g(p) - p \int_0^t F(p, u) du = 0 ,$$

that is,  $e^{pt} \times Z(p, t) \in F$ . In view of (4.85), (4.5) and (4.6), we have

$$1 \times P(p, t) = p \int_0^t P(p, u) du \in F .$$

Hence, by Theorem (3.24),  $P(p, t) \in F$ .

Now, let us suppose that, conversely,  $P(p, t) \in F$ . Then, in view of Theorem (4.22), the function (4.85) belongs to  $F$  and we have  $Z(p, t) \equiv 0$ . Hence,

$$g(p) = p \int_0^t F(p, u) du$$

and we obtain (4.82) and (4.83). •

(4.87) Theorem. If  $f(t) \in F$  and there exists a real non-negative number  $p_0$  such that the integral

$$(4.88) \quad g(p) := p \int_0^\infty e^{-pu} f(u) du$$

converges for  $p = p_0$  then

$$(4.89) \quad g(p) \in F \quad \text{and} \quad g(p) = f(t) .$$

Proof. It follows from the equality

$$(4.90) \quad \int_0^\infty e^{-pu} f(u+t) du = e^{-pt} \int_0^\infty e^{-pu} f(u) du$$

that this integral is convergent for  $p = p_0$  and any real  $t \geq 0$ . By virtue of the Fundamental Theorem for the Laplace Transformation ([1], p.35), the integral (4.90) is uniformly convergent with respect to real  $p > p_0$ . Thus, for every real  $T \geq 0$  and  $\varepsilon \geq 0$  there exists a real  $U > T$  such that

$$\left| \int_U^\infty e^{-pu} f(u+t) du \right| < \varepsilon , \quad \text{for real } p > p_0 .$$

Since we have for any real  $\omega$  from the interval  $0 \leq \omega \leq T+U$

$$\begin{aligned} & \left| \int_{U+T-\omega}^{\infty} e^{-\rho u} f(\omega+u) du \right| = \left| \int_{U+T}^{\infty} e^{-\rho(u-\omega)} f(\omega) du \right| = \\ & = \left| e^{-\rho(T-\omega)} \right| \left| \int_{U+T}^{\infty} e^{\rho(T-u)} f(u) du \right| \leq \left| \int_{U+T}^{\infty} e^{\rho(T-u)} f(u) du \right| = \\ & = \left| \int_U^{\infty} e^{-\rho(u-T)} f(u+T) du \right| < \varepsilon, \end{aligned}$$

we obtain for any natural number  $k$  and  $0 \leq t \leq T$

$$\begin{aligned} & \left| e^{-kp} \left( 1 \times \int_t^{\infty} e^{-\rho(t-u)} f(u) du \right) \right| = \left| p e^{-kp} \int_0^t \int_u^{\infty} e^{-\rho(u-v)} f(v) dv du \right| = \\ & = \left| p e^{-kp} \int_0^t \int_u^{\infty} e^{-\rho(u+v)} f(u+v) du dv \right| \leq \left| p e^{-kp} \int_0^t \int_0^{U+T-u} e^{-\rho(u+v)} f(u+v) du dv \right| + \\ & + \left| p e^{-kp} \int_0^t \int_{U+T-u}^{\infty} e^{-\rho(u+v)} f(u+v) du dv \right| du \leq \\ & \leq \left| p e^{-kp} \int_0^t \int_u^{U+T} e^{-\rho(u+v)} f(u+v) du dv \right| du + \left| p e^{-kp} \right| T \varepsilon \leq \\ & \leq \left| p e^{-kp} \right| \int_0^T \int_u^{U+T} e^{-\rho(u+v)} |f(u+v)| du dv + \left| p e^{-kp} \right| T \varepsilon \leq \left| p e^{-kp} \right| T(M+\varepsilon), \end{aligned}$$

where  $M$  is any positive number satisfying the condition

$$\int_0^{U+T} |f(u)| du < M.$$

It follows that

$$e^{-kp} \left( 1 \times \int_t^{\infty} e^{-\rho(t-u)} f(u) du \right) \underset{p \rightarrow \infty}{\Rightarrow} 0,$$

and, consequently, in view of Theorem (3.23),

$$(4.91) \quad F(p, t) := p \int_t^\infty e^{p(t-u)} f(u) du \in F.$$

By Theorem (4.81), we obtain (4.89). •

(4.92) Theorem. If  $F(p, t)$ ,  $t F(p, t)$ ,  $\partial/\partial t F(p, t)$ ,  $\partial/\partial t(t F(p, t)) \in F$  then

$$(4.93) \quad t \partial/\partial t F(p, t) \equiv (pt-1) F(p, t).$$

Proof. By virtue of (4.76) we have

$$\partial/\partial t(t F(p, t)) \equiv pt F(p, t).$$

Substituting here  $\partial/\partial t(t F(p, t)) = F(p, t) + t \partial/\partial t F(p, t)$ , we obtain (4.93). •

## V. Examples of applications

All problems solvable in the operational calculus or in the theory of the Laplace-Transformation can be solved also on the basis of the theory presented here. We shall show some examples.

First, we introduce one definition more.

(5.1) Definition. The function  $g(p) \in F$  will be called a **transform** of a function  $F(p, t) \in F$ , iff

$$(5.2) \quad g(p) \equiv F(p, t) . \quad \bullet$$

If there exists a transform (5.2) of a function  $F(p, t)$ , it is not unique, because we have also

$$g(p) + h(p) \equiv F(p, t) ,$$

where  $h(p) \in F$  is any function satisfying the congruence

$$h(p) \equiv 0 .$$

If  $F(p, t)$  satisfies the assumptions of Theorem (4.81), its transform can be found by means of the formula (4.82), which is a generalization of the Laplace-Carson transformation for functions of two variables  $p$  and  $t$ . But in many cases it is also possible to find this transform in another simpler way, supposing, of course, that such a transform exists and is a feasible function.

(5.3) Example. We shall find transforms of the following functions :

$$t^k/k! \quad (\text{$k$ positive integer}),$$

$$e^{kt} \quad (\text{$k$ real and $k \neq 0$}),$$

$$\sin \omega t \quad \text{and} \quad \cos \omega t \quad (\omega \text{ real and } \omega \neq 0),$$

$$e^{pt} \quad (\text{$k < r$}),$$

$$e^{-kr't} \quad (\text{$k$ and $r$ positive}),$$

$$\sin pt \quad \text{and} \quad \cos pt.$$

It follows from (4.79) that

$$\frac{d}{dt} e^{kt} (t^k/k!) = t = p^k (t^k/k!),$$

and hence

$$(5.4) \quad t^k/k! \equiv 1/p^k.$$

By virtue of (4.21) we have

$$e^{kt} = p \int_0^t e^{ku} du = p/k e^{kt} - p/k$$

and hence

$$(5.5) \quad e^{kt} = \frac{p}{p-k}.$$

In a similar way we obtain

$$\sin \omega t \equiv p \int_0^t \sin \omega u du = -p/\omega \cos \omega t + p/\omega,$$

$$\cos \omega t \equiv p \int_0^t \cos \omega u du = p/\omega \sin \omega t,$$

and hence

$$(5.6) \quad \sin \omega t \equiv \frac{p\omega}{p^2 + \omega^2},$$

$$(5.7) \quad \cos \omega t \equiv \frac{p^2}{p^2 + \omega^2}.$$

Now, we have

$$e^{kpt} = p \int_0^t e^{ku} du = 1/k e^{kt} - 1/k$$

and hence, for  $k < t$ ,

$$e^{kpt} = \frac{1}{t-k}$$

Similarly

$$e^{-kp^f t} = p \int_0^t e^{-kp^f u} du = -\frac{1}{p^{f-1} k} e^{-kp^f t} + \frac{1}{p^{f-1} k}$$

and hence

$$e^{-kp^f t} = \frac{1}{k p^{f-1} + t}$$

We find the transforms of  $\sin pt$  and  $\cos pt$  analogously to (5.6) and (5.7) and we obtain

$$\sin pt = \cos pt = 1/2 = e^{-pt} .$$

(5.8) Example. Let us solve the following differential equation

$$(5.9) \quad x''' - x'' + 4x' - 4x = 12t - 32$$

with initial conditions

$$x(0) = 5, \quad x'(0) = -2, \quad x''(0) = 5 .$$

By virtue of Theorem (4.78) we have

$$x' = px - 5p ,$$

$$x'' = p^2 x - 5p^2 + 2p ,$$

$$x''' = p^3 x - 5p^3 + 2p^2 - 5p ,$$

and by (5.4)

$$(5.10) \quad t = t/p .$$

Thus, by (5.9), we obtain

$$(\rho^3 x - 5\rho^3 + 2\rho^2 - 5\rho) - (\rho^2 x - 5\rho^2 + 2\rho) + 4(\rho x - 5\rho) - 4x = \\ \equiv 12/\rho - 32$$

i.e. after arrangement

$$\rho(\rho^3 - \rho^2 + 4\rho - 4)x = 5\rho^4 - 7\rho^3 + 27\rho^2 - 32\rho + 12$$

and hence

$$x = \frac{5\rho^4 - 7\rho^3 + 27\rho^2 - 32\rho + 12}{\rho(\rho^3 - \rho^2 + 4\rho - 4)} = \\ = 5 - \frac{3}{\rho} + \frac{\rho}{\rho-1} - \frac{\rho^2}{\rho^2+4}$$

By virtue of (5.10), (5.5) and (5.7) we obtain

$$x = 5 - 3t + e^t - \cos 2t$$

and by Theorem (4.32)

$$x = 5 - 3t + e^t - \cos 2t .$$

(5.11) Example. Let us find the general solution of the following differential equation

$$(5.12) \quad (t^3 - 2t + 1)x'''' + (-t^3 + 9t^2 + 2t - 7)x''' + (-8t^2 + 18t + 4)x'' + \\ + (-8t + 8)x' = e^{-t} .$$

Introducing

$$(5.13) \quad y = (t^3 - 2t + 1)x$$

we obtain

$$y' = (t^3 - 2t + 1)x' + (3t^2 - 2)x ,$$

$$y'' = (t^3 - 2t + 1)x'' + (6t^2 - 4)x' + 8tx ,$$

$$y''' = (t^3 - 2t + 1)x'''' + (9t^2 - 8)x''' + 18tx' + 8x$$

and we can write (5.12) in the form

$$y''' - y'' = e^{-t}.$$

In a similar way as in the previous example we obtain

$$y \equiv \frac{\rho^4 y_0 + \rho^3 y'_0 + \rho^2 (y''_0 - y_0) + \rho (y''_0 - y'_0 + 1)}{(\rho + 1)(\rho^3 - \rho^2)}$$

where

$$y_0 = y(0), \quad y'_0 = y'(0), \quad y''_0 = y''(0).$$

Hence

$$y \equiv R + \frac{\delta}{\rho} + \frac{C\rho}{\rho - 1} - \frac{1}{2} \frac{\rho}{\rho + 1}$$

where

$$R = -y''_0 + y_0, \quad \delta = -y''_0 + y'_0 - 1, \quad C = y''_0 + 1/2$$

and by (5.10) and (5.5)

$$y \equiv R + \delta t + C e^t - 1/2 e^{-t}$$

i.e. by virtue of Theorem (4.32)

$$y = R + \delta t + C e^t - 1/2 e^{-t}.$$

Hence, by (5.13)

$$x = \frac{R + \delta t + C e^t - 1/2 e^{-t}}{t^3 - 2t + 1}.$$

## VI. The subring $\mathcal{A}$

(6.1) Definition. We say that a complex function  $F(p, t)$  of a complex variable  $p$  and a real variable  $t$  belongs to the class  $\mathcal{A}$  iff there exists a real non-negative number  $p_0$  such that :

1<sup>o</sup> for almost every  $t \geq 0$  the function  $g_t(p) := F(p, t)$  is analytic in the half-plane  $\operatorname{Re} p \geq p_0$ ,

2<sup>o</sup> for every  $p$  with  $\operatorname{Re} p \geq p_0$  the functions  $F(p, t)$ ,  $d/dp F(p, t)$ ,  $d^2/dp^2 F(p, t)$ , ... are measurable with respect to  $t$  in the Lebesgue sense on every finite interval  $0 \leq t \leq T$ , and

3<sup>o</sup> there exist a real positive number  $\kappa$  and functions  $h_0(t)$ ,  $h_1(t)$ ,  $h_2(t)$ , ..., defined for almost every  $t \geq 0$  and integrable in the Lebesgue sense on every finite interval  $0 \leq t \leq T$ , such that for every  $p$  with  $\operatorname{Re} p \geq p_0$  and almost every  $t \geq 0$

$$(6.2) \quad |d^a/dp^a F(p, t)| \leq e^{\kappa p} h_a(t) \quad \text{for } a = 0, 1, 2, \dots,$$

where we put

$$d/dp^0 F(p, t) := F(p, t).$$

(6.3) Theorem. If  $F(p, t) \in \mathcal{A}$  then for every  $p$  with  $\operatorname{Re} p \geq p_0$  the functions  $F(p, t)$ ,  $d/dp F(p, t)$ ,  $d^2/dp^2 F(p, t)$ , ... are integrable with respect to  $t$  in the Lebesgue sense on every finite interval  $0 \leq t \leq T$ .

Proof. The theorem follows from the conditions 2<sup>o</sup> and 3<sup>o</sup> in Definition (6.1). •

(6.4) Theorem. If a function  $F(p, t)$  satisfies the condition (6.2) for  $n = 0, 1, \dots, n-1$  (or  $n = 0, 1, 2, \dots$ ) then

$$(6.5) \quad \frac{d^n}{dp^n} \int_0^t F(p, u) du = \int_0^t \frac{d^n}{dp^n} F(p, u) du \quad \text{for } n = 0, 1, \dots, n \\ (\text{or } n = 0, 1, 2, \dots)$$

Proof. We prove the theorem by induction. It is obvious that (6.5) is true for  $n = 0$ . Thus, let us suppose that it is true for  $n = r$ , i.e.,

$$(6.6) \quad \frac{d^r}{dp^r} \int_0^t F(p, u) du = \int_0^t \frac{d^r}{dp^r} F(p, u) du .$$

By virtue of (6.2) we have

$$|\frac{d}{dp} (e^{-kp} \frac{d^r}{dp^r} F(p, t))| = |-k e^{-kp} \frac{d^r}{dp^r} F(p, t) + e^{-kp} \frac{d^{r+1}}{dp^{r+1}} F(p, t)| \leq \\ \leq k |\frac{d^r}{dp^r} F(p, t)| + |e^{-kp} \frac{d^{r+1}}{dp^{r+1}} F(p, t)| \leq \\ \leq k h_r(t) + h_{r+1}(t)$$

and, therefore, we have

$$(6.7) \quad \frac{d}{dp} \int_0^t e^{-kp} \frac{d^r}{dp^r} F(p, u) du = \int_0^t \frac{d}{dp} (e^{-kp} \frac{d^r}{dp^r} F(p, u)) du .$$

In view of (6.6) we have

$$\begin{aligned} & \frac{d}{dp} \int_0^t e^{-kp} \frac{d^r}{dp^r} F(p, u) du = \frac{d}{dp} \left( e^{-kp} \int_0^t \frac{d^r}{dp^r} F(p, u) du \right) = \\ &= \frac{d}{dp} \left( e^{-kp} \frac{d^r}{dp^r} \int_0^t F(p, u) du \right) = \\ &= -k e^{-kp} \frac{d^r}{dp^r} \int_0^t F(p, u) du + e^{-kp} \frac{d^{r+1}}{dp^{r+1}} \int_0^t F(p, u) du = \\ &= -k e^{-kp} \frac{d^r}{dp^r} \int_0^t F(p, u) du + e^{-kp} \frac{d^{r+1}}{dp^{r+1}} \int_0^t F(p, u) du \end{aligned}$$

and

$$\begin{aligned} & \int_0^t \frac{d}{dp} (e^{-kp} \frac{d^r}{dp^r} F(p, u)) du = \\ &= -k e^{-kp} \int_0^t \frac{d^r}{dp^r} F(p, u) du + e^{-kp} \int_0^t \frac{d^{r+1}}{dp^{r+1}} F(p, u) du . \end{aligned}$$

Substituting this into (6.7) and multiplying both sides by  $e^{kp}$ , we obtain

$$s + 1/\alpha p + 1 \int_0^t F(p, u) du = \int_0^t s + 1/\alpha p + 1 F(p, u) du.$$

We see that if (6.5) is true for  $s = r$  then it is true for  $s = r+1$ , too. This completes the proof. •

(6.8) Theorem. The class  $\mathcal{B}$  forms a subring of the ring  $F$ .

Proof. It follows from Definition (2.1) and (6.1) that  $\mathcal{B} \subset S$  and from (6.2), for any  $F(p, t) \in \mathcal{B}$ ,

$$\left| e^{-\alpha p} \int_0^t F(p, u) du \right| \leq e^{-\alpha p} \int_0^t |F(p, u)| du \leq e^{-(q+k)p} \int_0^t h(u) du$$

which means that for any  $q > k$

$$e^{-\alpha p} \int_0^t F(p, u) du \underset{p \rightarrow \infty}{\Rightarrow} 0.$$

By Theorem (3.3),  $F(p, t) \in F$ .

This means that  $\mathcal{B} \subset F$  and it remains to be shown that  $\mathcal{B}$  forms a ring with respect to addition and the convolution (2.19).

Let us suppose that  $F_1(p, t), F_2(p, t) \in \mathcal{B}$  and

$$(6.9) \quad \left| \frac{d^n}{dt^n} F_j(p, t) \right| \leq \alpha^n p^{n-j} n_{j,n}(t), \quad j = 1, 2; \quad n = 0, 1, 2, \dots,$$

for every  $p$  with  $\operatorname{Re} p \geq p_j$  and almost every  $t \geq 0$ . Then we have (6.9) also for every  $p$  with  $\operatorname{Re} p \geq p_0 := \max(p_1, p_2)$  and almost every  $t \geq 0$ .

It is obvious that  $\mathcal{B}$  forms a group with respect to addition, because the functions  $F_1(p, t) \pm F_2(p, t)$  are analytic with respect to  $p$  in the half-plane  $\operatorname{Re} p \geq p_0$  and, moreover, for every  $p$  with  $\operatorname{Re} p \geq p_0$  the functions  $F_1(p, t) \pm F_2(p, t)$ ,  $d/dt(F_1(p, t) \pm F_2(p, t))$ ,  $d^2/dt^2(F_1(p, t) \pm F_2(p, t))$ , ... are integrable with respect to  $t$  in the Lebesgue sense on every finite interval  $0 \leq t \leq T$  and

$$\left| \frac{d^n}{dt^n} (F_1(p, t) \pm F_2(p, t)) \right| \leq e^{\alpha p} (n_{1,n}(t) + n_{2,n}(t)) \quad \text{for } n = 0, 1, 2, \dots$$

where  $k := \max(k_1, k_2)$ .

We have for every  $p$  with  $\operatorname{Re} p \geq p_0$ , almost every  $t \geq 0$  and almost every  $u$  from the interval  $0 \leq u \leq t$

$$\begin{aligned}
 |\partial/\partial p (\rho F_1(p, t-u) F_2(p, u))| &= |F_1(p, t-u) F_2(p, u) + \rho \partial/\partial p F_1(p, t-u) F_2(p, u) + \\
 &\quad + \rho F_1(p, t-u) \partial/\partial p F_2(p, u)| \leq \\
 &\leq e^{(k_1+k_2+1)p} (h_{10}(t-u) h_{20}(u) + h_{11}(t-u) h_{20}(u) + h_{10}(t-u) h_{21}(u)) \leq \\
 &\leq e^{(k_1+k_2+1)p} (h_{10}(t-u) + h_{11}(t-u)) (h_{20}(u) + h_{21}(u))
 \end{aligned}$$

and, by virtue of Theorem (6.4),

$$\begin{aligned}
 \partial/\partial p (F_1(p, t) \times F_2(p, t)) &= \partial/\partial p \left( \rho \int_0^t F_1(p, t-u) F_2(p, u) du \right) = \\
 &= \int_0^t \partial/\partial p (\rho F_1(p, t-u) F_2(p, u)) du,
 \end{aligned}$$

which means that the function  $F_1(p, t) \times F_2(p, t)$  is analytic with respect to  $p$  in the half-plane  $\operatorname{Re} p > p_0$ . It follows also that the functions  $F_1(p, t) \times F_2(p, t)$ ,  $\partial/\partial p (F_1(p, t) \times F_2(p, t))$ ,  $\partial^2/\partial p^2 (F_1(p, t) \times F_2(p, t))$ , ... are integrable with respect to  $t$  in the Lebesgue sense on every finite interval  $0 \leq t \leq T$ .

On the other hand, we have for  $n = 0, 1, 2, \dots$

$$\begin{aligned}
 \partial^n/\partial p^n (F_1(p, t) \times F_2(p, t)) &= \partial^n/\partial p^n \left( \rho \int_0^t F_1(p, t-u) F_2(p, u) du \right) = \\
 &= \rho \sum_{j=0}^{n-1} \binom{n}{j} \int_0^t \partial^{j+1}/\partial p^{j+1} F_1(p, t-u) \partial^{n-j-1}/\partial p^{n-j-1} F_2(p, u) du + \\
 &\quad + \rho \sum_{j=0}^n \binom{n}{j} \int_0^t \partial^{j+1}/\partial p^{j+1} F_1(p, t-u) \partial^{n-j-1}/\partial p^{n-j-1} F_2(p, u) du
 \end{aligned}$$

and, by (6.9),

$$\begin{aligned}
 |\partial^n/\partial p^n (F_1(p, t) \times F_2(p, t))| &\leq \\
 &\leq e^{(k_1+k_2+1)p} \left( \rho \sum_{j=0}^{n-1} \binom{n}{j} \int_0^t h_{1j}(t-u) h_{2,n-1-j}(u) du + \sum_{j=0}^n \binom{n}{j} \int_0^t h_{1j}(t-u) h_{2,n-j}(u) du \right).
 \end{aligned}$$

It follows that for any  $F_1(p, t), F_2(p, t) \in A$  we have  $F_1(p, t) \times F_2(p, t) \in A$ . This completes the proof. •

(6.10) Definition. A function  $F(p, t)$  will be called **superfeasible** iff  $F(p, t) \in A$ . •

(6.11) Theorem. We have  $f(t) \in A$  iff  $f(t) \in S$ , that is, iff  $f(t)$  is defined for almost all  $t \geq 0$  and integrable in the Lebesgue sense on every finite interval  $0 \leq t \leq T$ .

Proof. If  $f(t) \in A$  then, by Theorems (6.8) and (3.13),  $f(t) \in S$ . Conversely, if  $f(t) \in S$  then, by Definition (6.1),  $f(t) \in A$ . •

(6.12) Theorem. For any real  $r$  we have  $e^{rt} \in A$ .

Proof. The theorem follows from Definition (6.1). •

(6.13) Theorem. For any real  $r$  we have  $e^{rp} \in A$ .

Proof. The theorem follows from Definition (6.1). •

(6.14) Theorem. If  $F(p, t) \in A$  then for any function  $f(t) \in S$  we have  $f(t) \times F(p, t) \in A$ .

Proof. The theorem follows from Theorems (6.8) and (6.11). •

(6.15) Theorem. If  $F(p, t) \in A$  and  $g(p) \in A$  then

$$g(p) F(p, t) \in A.$$

Proof. The function  $g(p) F(p, t)$  satisfies the conditions 1 $\alpha$  and 2 $\alpha$  from Definition (6.1). Moreover, we have, according to the condition 3 $\alpha$ ,

$$\left| \frac{d^m}{dp^m} g(p) \right| \leq c_m e^{kp},$$

$$\left| \frac{d^m}{dp^m} F(p, t) \right| \leq b_m e^{qt} h(t) \quad \text{for } m = 0, 1, 2, \dots$$

where  $k$ ,  $q$ ,  $c_m$  and  $b_m$  are real positive numbers. Hence

$$\left| \frac{d^m}{dp^m} (g(p) F(p, t)) \right| =$$

$$= \left| \frac{d^m}{dp^m} g(p) \cdot F(p, t) + g(p) \cdot \frac{d^m}{dp^m} F(p, t) \right| \leq$$

$$\begin{aligned} &\leq \left| \frac{\partial^k}{\partial p^k} g(p) \right| |F(p, t)| + |g(p)| \left| \frac{\partial^k}{\partial p^k} F(p, t) \right| \leq \\ &\leq c_k e^{kp} b_0 e^{qt} + c_0 e^{kp} b_k e^{qt} = \\ &= (c_k b_0 + c_0 b_k) e^{(k+q)p} \quad \text{for } k = 0, 1, 2, \dots \end{aligned}$$

which means that the function  $g(p) F(p, t)$  satisfies also the condition 3° from Definition (6.1). It follows that  $g(p) F(p, t) \in A$ . •

(6.16) Theorem. For any real  $r$  we have

$$F(p, t) \in A \quad \text{iff} \quad p^r F(p, t) \in A .$$

Proof. By Theorems (6.12) and (6.15), if  $F(p, t) \in A$  then  $p^r F(p, t) \in A$ .

Conversely, if  $p^r F(p, t) \in A$  then, by virtue of the first part of our theorem,  $F(p, t) = p^{-r} p^r F(p, t) \in A$ . •

(6.17) Theorem. If  $F(p, t) \in A$  then also

$$\int_0^t F(p, u) du \in A \quad \text{and} \quad p \int_0^t F(p, u) du \in A .$$

Proof. Let  $f(t) := t$ . Then, by virtue of Theorems (6.11) and (6.14),

$$p \int_0^t F(p, u) du = t \times F(p, t) = f(t) \times F(p, t) \in A .$$

Since

$$\int_0^t F(p, u) du = 1/p \int_0^t p F(p, u) du$$

and  $1/p = p^{-1} \in A$ , we obtain, by Theorem (6.15),

$$\int_0^t F(p, u) du \in A .$$

This completes the proof. •

(6.18) Definition. The class  $Z_{\mathcal{B}}$  is the set of all functions  $Z(p, t) \in \mathcal{B}$  satisfying the condition

(6.19)  $e^{pt} \times Z(p, t) \in \mathcal{B}$ .

(6.20) Theorem. The class  $Z_{\mathcal{B}}$  forms an ideal of the subring  $\mathcal{B}$ .

Proof. We prove the theorem analogously to Theorem (4.7). •

(6.21) Definition. The  $\mathcal{B}$ -distribution Ring or, simply, the ring  $\mathcal{D}_{\mathcal{B}}$  is the residue class ring  $\mathcal{B}/Z_{\mathcal{B}}$  into which the ideal  $Z_{\mathcal{B}}$  divides the subring  $\mathcal{B}$ . •

(6.22) Definition. An  $\mathcal{B}$ -distribution is any residue class belonging to the ring  $\mathcal{D}_{\mathcal{B}}$ . •

(6.23) Definition. Each function  $F(p, t) \in \mathcal{B}$  belonging to an  $\mathcal{B}$ -distribution  $\mathcal{B}$  is to be called a representative of this  $\mathcal{B}$ -distribution, which is written in the form

$$\mathcal{F} = \{ F(p, t) \} .$$

(6.24) Definition. Any functions  $F_1(p, t), F_2(p, t) \in \mathcal{B}$  are said to be  $\mathcal{B}$ -congruent iff they are representatives of the same  $\mathcal{B}$ -distribution and in such a case we write

(6.25)  $F_1(p, t) \equiv F_2(p, t)$ .

It follows that (6.25) holds iff  $F_1(p, t) - F_2(p, t) \in Z_{\mathcal{B}}$ . The congruence  $F(p, t) \equiv 0$  means that  $F(p, t) \in Z_{\mathcal{B}}$ .

(6.26) Theorem. The relation (6.25) is a congruence modulo  $Z_{\mathfrak{A}}$ , that is, it has properties analogous to (4.14)-(4.19).

Proof. All the above properties are well-known properties of any residue class ring. •

(6.27) Theorem. For every  $F(p, t) \in \mathfrak{A}$  we have

$$(6.28) \quad F(p, t) \equiv p \int_0^t F(p, u) du .$$

Proof. We prove the theorem analogously to Theorem (4.20). •

(6.29) Theorem. A function  $Z(p, t) \in \mathfrak{A}$  belongs to the ideal  $Z_{\mathfrak{A}}$  iff there exists a function  $F(p, t) \in \mathfrak{A}$  such that

$$(6.30) \quad Z(p, t) = F(p, t) - 1 \times F(p, t) = F(p, t) - p \int_0^t F(p, u) du .$$

Proof. We prove the theorem analogously to Theorem (4.22). •

(6.31) Theorem. If  $g(p) \in \mathfrak{A}$  and  $F(p, t) \in \mathfrak{A}$  then

$$(6.32) \quad g(p) \times F(p, t) \equiv g(p) F(p, t) .$$

Proof. We prove the theorem analogously to Theorem (4.24). •

(6.33) Theorem. If  $F_1(p, t) \equiv F_2(p, t)$  and  $g(p) \in \mathfrak{A}$  then

$$(6.34) \quad g(p) F_1(p, t) \equiv g(p) F_2(p, t) .$$

Proof. We prove the theorem analogously to Theorem (4.26). •

(6.35) Theorem. If  $F(p, t) \equiv 0$  then

$$(6.36) \quad p \int_0^t F(p, u) du \equiv 0 .$$

Proof. The theorem follows from Theorems (6.17) and (6.27). •

(6.37) Theorem. If  $F(p, t) \equiv 0$  then

$$(6.38) \quad \int_0^t F(p, u) du \equiv 0 .$$

Proof. We prove the theorem analogously to Theorem (4.30). •

(6.39) Theorem. If  $F_1(p, t) \equiv F_2(p, t)$  then  $F_1(p, t) = F_2(p, t)$ .

Proof. The theorem follows from Theorem (6.8) and Definitions (4.1) and (6.18), because  $F_1(p, t) - F_2(p, t) \in Zg \subset Z$ . •

(6.40) Theorem. For every function  $r(t) \in S$  we have

$$(6.41) \quad r(t) \equiv 0$$

iff

$$(6.42) \quad r(t) = 0 \quad \text{a.e.}$$

Proof. If we have (6.41) then, by virtue of Theorems (6.39) and (4.32) we obtain (6.42). Conversely, if we have (6.42) then

$$aP^t \times r(t) = 0 \in A ,$$

which is equivalent to (6.41). •

(6.43) Theorem. The  $\#$ -congruence

$$f_1(t) \equiv f_2(t)$$

holds iff

$$f_1(t) = f_2(t) \quad \text{a.e.}$$

Proof. We obtain this immediately, when replacing the function  $f(t)$  in Theorem (6.40) by  $f_1(t) - f_2(t)$ . •

(6.44) Theorem. If  $F(p, t) \in \mathcal{B}$  and for any real  $r$

$$(6.45) \quad F(p, t) = 0 \quad \begin{cases} \text{for } 0 \leq t \leq r \text{ when } r > 0, \\ \text{for } r \leq t \leq 0 \text{ when } r < 0, \end{cases}$$

then

$$(6.46) \quad G(p, t) := F(p, t+r) \in \mathcal{B}$$

and

$$(6.47) \quad G(p, t) = 0 \quad \begin{cases} \text{for } -r \leq t \leq 0 \text{ when } r > 0, \\ \text{for } 0 \leq t \leq -r \text{ when } r < 0. \end{cases}$$

Proof. By Theorems (6.8) and (4.57), we have  $G(p, t) \in \mathcal{F}$ . Since  $F(p, t) \in \mathcal{B}$ , the function  $G(p, t)$  satisfies the conditions 1<sup>o</sup> and 2<sup>o</sup> from Definition (6.1). But  $F(p, t)$  satisfies the condition (6.2), which implies

$$(6.48) \quad | \frac{d^n}{dt^n} G(p, t) | \leq e^{kp} h_n(t+r) \quad \text{for } n = 0, 1, 2, \dots,$$

and  $G(p, t) \in \mathcal{B}$ . •

(6.49) Theorem. If, for any real  $r$ ,  $F(p, t) \in \mathcal{B}$  is a function satisfying the condition (6.45) and  $G(p, t)$  is defined by (6.46) then

$$(6.50) \quad G(p, t) \equiv F(p, t) e^{rp}.$$

Proof. Let

$$K(p, t) := \int_t^{t+r} e^{pu} F(p, u) du$$

and let us assume that the function  $F(p, t)$  satisfies the condition (6.2). First, we shall show that

$$(6.51) \quad \frac{\partial^s}{\partial p^s} K(p, t) = \int_t^{t+r} \frac{\partial^s}{\partial p^s} (e^{p(t-u)} F(p, u)) du .$$

This is true for  $s=0$ . Let us suppose that it is true for  $s=s$ , that is,

$$\frac{\partial^s}{\partial p^s} K(p, t) = \int_t^{t+r} \frac{\partial^s}{\partial p^s} (e^{p(t-u)} F(p, u)) du .$$

We have for  $t \leq u \leq t+r$

$$\begin{aligned} & |\frac{\partial}{\partial p} (e^{-kp} \frac{\partial^s}{\partial p^s} (e^{p(t-u)} F(p, u)))| = \\ & = | -k e^{-kp} \frac{\partial^s}{\partial p^s} (e^{p(t-u)} F(p, u)) + e^{-kp} \frac{\partial^{s+1}}{\partial p^{s+1}} (e^{p(t-u)} F(p, u)) | = \\ & = | -k e^{-kp} \sum_{j=0}^s \binom{s}{j} (t-u)^j e^{p(t-u)} \frac{\partial^{s-j}}{\partial p^{s-j}} F(p, u) + \\ & \quad + e^{-kp} \sum_{j=0}^{s+1} \binom{s+1}{j} (t-u)^j e^{p(t-u)} \frac{\partial^{s+1-j}}{\partial p^{s+1-j}} F(p, u) | \leq \\ & \leq k \sum_{j=0}^s \binom{s}{j} n_j h_{s-j}(t) + \sum_{j=0}^{s+1} \binom{s+1}{j} n_j h_{s+1-j}(t) = h(t) , \end{aligned}$$

Hence we have

$$\begin{aligned} (6.52) \quad & \frac{\partial}{\partial p} \int_t^{t+r} e^{-kp} \frac{\partial^s}{\partial p^s} (e^{p(t-u)} F(p, u)) du = \\ & = \int_t^{t+r} \frac{\partial}{\partial p} (e^{-kp} \frac{\partial^s}{\partial p^s} (e^{p(t-u)} F(p, u))) du . \end{aligned}$$

Since

$$\begin{aligned} & \frac{\partial}{\partial p} \int_t^{t+r} e^{-kp} \frac{\partial^s}{\partial p^s} (e^{p(t-u)} F(p, u)) du = \\ & = -k e^{-kp} \int_t^{t+r} \frac{\partial^s}{\partial p^s} (e^{p(t-u)} F(p, u)) du + \\ & \quad + e^{-kp} \frac{\partial}{\partial p} \int_t^{t+r} \frac{\partial^s}{\partial p^s} (e^{p(t-u)} F(p, u)) du \end{aligned}$$

and

$$\begin{aligned} & \int_t^{t+r} \frac{\partial}{\partial p} (e^{-kp} \frac{\partial^s}{\partial p^s} (e^{p(t-u)} F(p, u)) du = \\ & = -k e^{-kp} \int_t^{t+r} \frac{\partial^s}{\partial p^s} (e^{p(t-u)} F(p, u)) du + \\ & \quad + e^{-kp} \int_t^{t+r} \frac{\partial^{s+1}}{\partial p^{s+1}} (e^{p(t-u)} F(p, u)) du , \end{aligned}$$

we obtain from (6.52)

$$\begin{aligned} \partial^r + 1/\partial p^r + 1 K(p, t) &= \partial/\partial p \int_t^{t+r} (\partial^r/\partial p^r (e^{-p(t-u)} F(p, u))) du = \\ &= \int_t^{t+r} (\partial^r + 1/\partial p^r + 1 (e^{-p(t-u)} F(p, u))) du . \end{aligned}$$

We see that if (6.51) is true for  $\alpha = s$  then it is also true for  $\alpha = s+1$ . Thus (6.51) is true for  $\alpha = 0, 1, 2, \dots$ .

It follows from (6.51) that the function  $K(p, t)$  satisfies the conditions 1° and 2° from Definition (6.1). We have further

$$\begin{aligned} |\partial^r/\partial p^r K(p, t)| &= \left| \int_t^{t+r} \partial^r/\partial p^r (e^{-p(t-u)} F(p, u)) du \right| \leq \\ &\leq \int_t^{t+r} \left| \partial^r/\partial p^r (e^{-p(t-u)} F(p, u)) \right| du = \\ &= \int_t^{t+r} \left| \sum_{j=0}^s \binom{s}{j} (t-u)^j e^{-p(t-u)} \partial^{r-j}/\partial p^{r-j} F(p, u) \right| du \leq \\ &\leq \int_t^{t+r} \sum_{j=0}^s \binom{s}{j} r^j \left| \partial^{r-j}/\partial p^{r-j} F(p, u) \right| du \leq \\ &\leq \int_t^{t+r} \sum_{j=0}^s \binom{s}{j} r^j e^{-kp} h_{s-j}(u) du = \\ &= e^{-kp} b(t) , \end{aligned}$$

where

$$b(t) := \int_t^{t+r} \sum_{j=0}^s \binom{s}{j} r^j h_{s-j}(u) du .$$

Thus the function  $K(p, t)$  satisfies also the condition 3° from Definition (6.1) and we have  $K(p, t) \in \mathcal{B}$ .

Let

$$\begin{aligned} H(p, t) &:= e^{pt} \times (S(p, t) - F(p, t) e^{tp}) = \\ &= p \int_0^t e^{p(t-u)} (S(p, u) - F(p, u) e^{tp}) du = \\ &= p \int_0^t e^{p(t-u)} F(p, u+r) du - p \int_0^t e^{p(t-u+r)} F(p, u) du = \end{aligned}$$

$$\begin{aligned}
 &= p \int_{\rho}^{t+r} e^{\rho(t-u+r)} F(p, u) du - p \int_0^t e^{\rho(t-u+r)} F(p, u) du = \\
 &= p \int_{\rho}^{t+r} e^{\rho(t-u+r)} F(p, u) du - p \int_{\rho}^t e^{\rho(t-u+r)} F(p, u) du = \\
 &= p \int_{\rho}^{t+r} e^{\rho(t-u+r)} F(p, u) du = p e^{\rho t} K(p, t) .
 \end{aligned}$$

By virtue of Theorems (6.12), (6.13) and (6.15) we have  $H(p, t) \in \mathbb{R}$ . This means that

$$H(p, t) - F(p, t) e^{\rho t} \equiv 0 ,$$

which is equivalent to (6.50). •

(6.53) Theorem. If

$$F(p, t) \equiv g(p) ,$$

where  $F(p, t), g(p) \in \mathbb{R}$ , and we put  $F(p, t) = 0$  for  $-r \leq t < 0$ ,  $r$  being any real positive number, then

$$(6.54) \quad F(p, t-r) \equiv g(p) e^{-\rho r} .$$

Proof. The theorem follows from Theorem (6.49). •

(6.55) Theorem. If

$$F(p, t) \equiv g(p) ,$$

where  $F(p, t), g(p) \in \mathbb{R}$  and

$$(6.56) \quad F(p, t) = 0 \quad \text{for } 0 \leq t < r ,$$

$r$  being any real positive number, then

$$(6.57) \quad F(p, t+r) \equiv g(p) e^{\rho r} .$$

Proof. The theorem follows from Theorem (6.49). •

(6.58) Theorem. If  $F_1(p, t), F_2(p, t), g_1(p), g_2(p) \in \mathcal{A}$  and

$$(6.59) \quad F_1(p, t) \equiv g_1(p), \quad F_2(p, t) \equiv g_2(p),$$

then

$$(6.60) \quad F_1(p, t) \times F_2(p, t) \equiv g_1(p) g_2(p).$$

Proof. We prove the theorem analogously to Theorem (4.70). •

(6.61) Theorem. If  $f(t)$  is a measurable function bounded on every finite interval  $0 \leq t \leq T$  and  $F(p, t) \in \mathcal{A}$  then  $f(t) F(p, t) \in \mathcal{A}$ .

Proof. The function  $f(t) F(p, t)$  satisfies all conditions of Definition (6.1) and hence it belongs to the subring  $\mathcal{A}$ . •

(6.62) Theorem. If  $F(p, t), d/dt F(p, t) \in \mathcal{A}$  then

$$(6.63) \quad d/dt F(p, t) \equiv p F(p, t) - p F(p, 0),$$

where also  $F(p, 0) \in \mathcal{A}$ .

Proof. If  $d/dt F(p, t) \in \mathcal{A}$  then, by virtue of Theorem (6.17), we have

$$(6.64) \quad \int_0^t d/du F(p, u) du = F(p, t) - F(p, 0) \in \mathcal{A}.$$

Hence  $F(p, 0) \in \mathcal{A}$  and

$$p \int_0^t d/du F(p, u) du = p F(p, t) - p F(p, 0).$$

By Theorem (6.27) we obtain (6.63). •

(6.65) Theorem. If  $F(p, t), d/dt F(p, t), \dots, d/dt^k F(p, t) \in \mathcal{A}$  then

$$(6.66) \quad d/dt^k F(p, t) \equiv p^k F(p, t) - p^k F(p, 0) - p^{k-1} F'(1)(p, 0) - \dots - p F^{(k-1)}(p, 0),$$

where

$$(6.67) \quad F^{(j)}(p, 0) := [\partial^j / \partial t^j F(p, t)]_{t=0} \in A \quad \text{for } j = 0, \dots, k-1.$$

Proof. We obtain (6.67) when applying Theorem (6.62)  $k$  times. •

$$(6.68) \quad \text{Theorem.} \quad \text{We have}$$

$$(6.69) \quad g(p) := p \int_0^\infty e^{-pu} F(p, u) du \in A$$

and

$$(6.70) \quad g(p) \equiv F(p, t)$$

where  $F(p, t) \in A$  iff

$$(6.71) \quad F(p, t) := p \int_t^\infty e^{p(t-u)} F(p, u) du \in A.$$

Proof. We prove the theorem analogously to Theorem (4.81). •

$$(6.72) \quad \text{Theorem.} \quad \text{If } F(p, t) \in A \text{ and } k \text{ is a real positive number satisfying the condition (6.2) then for almost every } t > k \text{ we have}$$

$$(6.73) \quad \lim_{p \rightarrow \infty} p e^{-pt} F(p, t) \rightarrow 0$$

Proof. We have, according to (6.2),

$$|F(p, t)| \leq e^{kp} h_0(t)$$

and for every  $t > k$

$$|p e^{-pt} F(p, t)| \leq e^{-(t-k)p} h_0(t) \xrightarrow[p \rightarrow \infty]{} 0.$$

Hence we obtain (6.73). •

(6.74) Theorem. If  $F(p, t)$ ,  $\partial/\partial t F(p, t)$ ,  $\partial/\partial t(t F(p, t)) \in A$  then

$$(6.75) \quad t \cdot \partial/\partial t F(p, t) \equiv (pt-1) F(p, t) .$$

Proof. We prove the theorem analogously to Theorem (4.92). •

(6.76) Theorem. We have

$$(6.77) \quad g(p) \equiv 0, \quad g(p) \in A \quad \text{iff} \quad e^{pt} g(p) \in A .$$

Proof. If  $g(p) \equiv 0$ ,  $g(p) \in A$  then, according to Definitions (6.24) and (6.18),

$$e^{pt} \times g(p) = g(p) (e^{pt-1}) \in A \quad \text{and hence} \quad e^{pt} g(p) \in A .$$

Thus, let us now suppose that

$$(6.78) \quad e^{pt} g(p) \in A .$$

According to (6.2), we have

$$(6.79) \quad | \partial^n/dp^n (e^{pt} g(p)) | \leq e^{kp} h_n(t) \quad \text{for } n = 0, 1, 2, \dots .$$

We shall show that there exist analogously functions  $H_0(t)$ ,  $H_1(t)$ ,  $H_2(t)$ , ... such that

$$(6.80) \quad | \partial^n/dp^n g(p) | \leq e^{kp} H_n(t) \quad \text{for } n = 0, 1, 2, \dots .$$

Since from (6.79) we have

$$| e^{pt} g(p) | \leq e^{kp} h_0(t) ,$$

it follows that

$$| g(p) | \leq e^{-pt} e^{kp} h_0(t) \leq e^{kp} h_0(t) ,$$

which means that (6.80) is true for  $n = 0$  with  $H_0(t) := h_0(t)$ . Now, let us suppose that (6.80) is true for  $n = 0, 1, \dots, n$ . Since, according to (6.79),

$$| \partial^{n+1}/dp^{n+1} (e^{pt} g(p)) | \leq e^{kp} h_{n+1}(t)$$

and

$$\begin{aligned} \sigma^{n+1}/\partial t^{n+1} (\sigma^t g(p)) &= \sum_{j=0}^{n+1} \binom{n+1}{j} \partial/\partial p^j \sigma^t \sigma^{n+1-j}/\partial t^{n+1-j} g(p) = \\ &= \sigma^t \sum_{j=0}^{n+1} \binom{n+1}{j} \sigma^{n+1-j}/\partial t^{n+1-j} g(p), \end{aligned}$$

that is,

$$\begin{aligned} \sigma^{n+1}/\partial t^{n+1} g(p) &= e^{-pt} \sigma^{n+1}/\partial t^{n+1} (e^{pt} g(p)) - \\ &- \sum_{j=1}^{n+1} \binom{n+1}{j} \sigma^j \sigma^{n+1-j}/\partial t^{n+1-j} g(p), \end{aligned}$$

we obtain

$$\begin{aligned} |\sigma^{n+1}/\partial t^{n+1} g(p)| &\leq |\sigma^{n+1}/\partial t^{n+1} (e^{pt} g(p))| + \\ &+ \sum_{j=1}^{n+1} \binom{n+1}{j} |\sigma^j \sigma^{n+1-j}/\partial t^{n+1-j} g(p)| \leq \\ &\leq e^{pt} h_{n+1}(t) + e^{pt} \sum_{j=1}^{n+1} \binom{n+1}{j} |\sigma^{n+1-j} H_{n+1-j}(t)| = \\ &= e^{pt} H_{n+1}(t), \end{aligned}$$

where

$$H_{n+1}(t) := h_{n+1}(t) + \sum_{j=1}^{n+1} \binom{n+1}{j} |\sigma^j H_{n+1-j}(t)|.$$

which means that (6.80) is also true for  $n = n+1$ . By induction, (6.80) is true for  $n = 0, 1, 2, \dots$ .

We see that the function  $g(p)$  satisfies the condition (6.2). Since  $g(p) = e^{-pt}(\sigma^t g(p))$ , it satisfies also the remaining conditions from Definition (6.1), which means that  $g(p) \in \mathcal{A}$ .

By virtue of (6.78) we have also

$$\sigma^t \times g(p) = g(p)(\sigma^{t-1}) \in \mathcal{A},$$

which means that  $g(p) \equiv 0$ . This completes the proof. •

## VII. The D-Derivative

(7.1) Definition. If  $F(p, t) \in A$  then the D-derivative of  $F(p, t)$  is the function

$$(7.2) \quad DF(p, t) := \frac{\partial}{\partial p} F(p, t) + (t/p - t) F(p, t) . \quad \bullet$$

(7.3) Theorem. If  $F(p, t) \in A$  then  $DF(p, t)$  exists and  $DF(p, t) \in A$ .

Proof. The theorem follows from Definitions (6.1) and (7.1) and Theorems (6.12) and (6.61).  $\bullet$

(7.4) Theorem. If  $F(p, t), G(p, t) \in A$  then

$$(7.5) \quad D(F(p, t) \pm G(p, t)) = DF(p, t) \pm DG(p, t) .$$

Proof. The theorem follows immediately from Definition (7.1).  $\bullet$

(7.6) Theorem. If  $F(p, t), G(p, t) \in A$  then

$$(7.7) \quad D(F(p, t) \times G(p, t)) = DF(p, t) \times G(p, t) + F(p, t) \times DG(p, t) .$$

Proof. We have, by virtue of Theorem (6.4),

$$\begin{aligned} D(F(p, t) \times G(p, t)) &= D\left(p \int_0^t F(p, t-u) G(p, u) du\right) = \\ &= \int_0^t F(p, t-u) G(p, u) du + p \int_0^t \frac{\partial}{\partial p} F(p, t-u) G(p, u) du + p \int_0^t F(p, t-u) \frac{\partial}{\partial p} G(p, u) du + \\ &+ \int_0^t F(p, t-u) \delta(p, u) du - pt \int_0^t F(p, t-u) G(p, u) du , \end{aligned}$$

and

$$\begin{aligned} DF(p, t) \times G(p, t) + F(p, t) \times D G(p, t) &= \\ &= (\frac{\partial}{\partial p} F(p, t) + 1/p F(p, t) - t F(p, t)) \times G(p, t) + \\ &+ F(p, t) \times (\frac{\partial}{\partial p} G(p, t) + 1/p G(p, t) - t G(p, t)) = \\ &= p \int_0^t \frac{\partial}{\partial p} F(p, t-u) G(p, u) du + \int_0^t F(p, t-u) \delta(p, u) du - p \int_0^t (t-u) F(p, t-u) G(p, u) du + \\ &+ p \int_0^t F(p, t-u) \frac{\partial}{\partial p} G(p, u) du + \int_0^t F(p, t-u) \delta(p, u) du - p \int_0^t u F(p, t-u) G(p, u) du . \end{aligned}$$

Hence we obtain (7.7). •

(7.8) Theorem. If  $F(p, t) \equiv 0$  then

$$DF(p, t) \equiv 0 .$$

Proof.  $F(p, t) \equiv 0$  means that

$$(7.9) \quad e^{pt} \times F(p, t) \in A .$$

On the other hand we have

$$\begin{aligned} D(e^{pt} \times F(p, t)) &= D\left(p \int_0^t e^{p(t-u)} F(p, u) du\right) = D\left(p e^{pt} \int_0^t e^{-pu} F(p, u) du\right) = \\ &= e^{pt} \int_0^t e^{-pu} F(p, u) du + p t e^{pt} \int_0^t e^{-pu} F(p, u) du - p e^{pt} \int_0^t e^{-pu} u F(p, u) du + \end{aligned}$$

$$+ p e^{pt} \int_0^t e^{-pu} \frac{d}{dp} F(p, u) du + e^{pt} \int_0^t e^{-pu} F(p, u) du - pt e^{pt} \int_0^t e^{-pu} F(p, u) du = \\ = 1/p (e^{pt} \times F(p, t)) + e^{pt} \times DF(p, t).$$

It follows from (7.9), by virtue of Theorems (7.3) and (6.16), that

$$D(e^{pt} \times F(p, t)) \in A \quad \text{and} \quad 1/p (e^{pt} \times F(p, t)) \in A.$$

Hence we have  $e^{pt} \times DF(p, t) \in A$ , which means that  $DF(p, t) \equiv 0$ . •

(7.10) Theorem. If  $F(p, t), G(p, t) \in A$  and

$$(7.11) \quad F(p, t) \equiv G(p, t),$$

then

$$(7.12) \quad DF(p, t) \equiv DG(p, t).$$

Proof. The theorem follows from Theorem (7.8) when we replace  $F(p, t)$  by  $F(p, t) - G(p, t)$ . •

(7.13) Theorem. If  $F(p, t), G(p, t) \in A$  and

$$(7.14) \quad F(p, t) \equiv G(p, t),$$

then

$$(7.15) \quad \frac{d}{dp} F(p, t) - t F(p, t) \equiv \frac{d}{dp} G(p, t) - t G(p, t).$$

Proof. It follows from (7.14), by virtue of Theorem (7.10), that

$$DF(p, t) \equiv DG(p, t),$$

that is,

$$(7.16) \quad \frac{d}{dp} F(p, t) + (1/p - t) F(p, t) \equiv \frac{d}{dp} G(p, t) + (1/p - t) G(p, t).$$

Since (7.14) implies, by Theorem (6.33),

$$1/p \ F(p, t) \equiv 1/p \ G(p, t),$$

we obtain (7.15) from (7.16). •

(7.17) Theorem. If  $f(t), g(p) \in A$  and

$$(7.18) \quad f(t) \equiv g(p),$$

then

$$(7.19) \quad t f(t) \equiv 1/p \ g(p) - d/dp \ g(p)$$

and

$$(7.20) \quad t \ d/dt \ f(t) \equiv -p \ d/dp \ g(p).$$

Proof. It follows from (7.15) that

$$(7.21) \quad t f(t) \equiv t g(p) - d/dp \ g(p).$$

Analogously to (5.4) we have

$$t \equiv 1/p$$

and, by Theorem (6.33),

$$(7.22) \quad t g(p) \equiv 1/p \ g(p).$$

From (7.21) and (7.22) we obtain (7.19).

It follows from (6.75) that

$$(7.23) \quad t \ d/dt \ f(t) \equiv (pt-1) f(t).$$

On the other hand, it follows from (7.19) that

$$(pt-1) f(t) \equiv g(p) - p \ d/dp \ g(p) - f(t).$$

In view of (7.18), we have

$$(7.24) \quad (pt-1) f(t) \equiv -p \frac{d}{dp} g(p) .$$

From (7.23) and (7.24) we obtain (7.20). •

(7.25) Theorem. For every  $F(p, t) \in \mathcal{A}$  we have

$$(7.26) \quad F(p, t) = 1/p e^{pt} \int p e^{-pt} DF(p, t) dp ,$$

where

$$H(p, t) := \int p e^{-pt} DF(p, t) dp$$

denotes the function satisfying the equation

$$(7.27) \quad \frac{d}{dp} H(p, t) = p e^{-pt} DF(p, t)$$

and the condition

$$(7.28) \quad \lim_{p \rightarrow \infty} H(p, t) \rightarrow 0 .$$

Proof. Let us introduce

$$(7.29) \quad H(p, t) := p e^{-pt} F(p, t) .$$

By virtue of Theorem (6.72), the function (7.29) satisfies the condition (7.28). Moreover, the function (7.29) satisfies the condition (7.27) too. Any other function satisfying (7.27) must have the form

$$H(p, t) := p e^{-pt} F(p, t) + f(t)$$

and satisfies the condition (7.28) iff  $f(t) = 0$ . Thus (7.29) is the only solution of (7.27) satisfying (7.28) and we have

$$F(p, t) = 1/p e^{pt} H(p, t) .$$

Hence we obtain (7.26). •

(7.30) Theorem. If  $F(p, t) \in \mathcal{A}$  then we have

$$(7.31) \quad DF(p, t) = 0 \quad \text{iff} \quad F(p, t) = 0 .$$

Proof. Substituting in (7.26)  $\mathbf{D}F(p, t) = 0$  we obtain  $F(p, t) = 0$ . Conversely, substituting in (7.2)  $F(p, t) = 0$ , we obtain  $\mathbf{D}F(p, t) = 0$ . •

(7.32) Theorem. For every  $F(p, t) \in \mathcal{A}$  we have

$$(7.33) \quad \mathbf{D}F(p, t) = 1/p e^{pt} \frac{d}{dp} (p e^{-pt} F(p, t)) ,$$

$$(7.34) \quad \frac{d}{dp} F(p, t) = p e^{-pt} \mathbf{D}(1/p e^{pt} F(p, t)) .$$

Proof. Substituting (7.29) into (7.27) and multiplying both sides by  $1/p e^{pt}$ , we obtain (7.33). Replacing  $1/p e^{pt} F(p, t)$  by  $F(p, t)$  in (7.33) and multiplying both sides by  $p e^{-pt}$ , we obtain (7.34). •

(7.35) Theorem. If  $F(p, t) \in \mathcal{A}$  and

$$(7.36) \quad \mathbf{D}F(p, t) \equiv 0$$

then there exists a function  $H(p, t) \in \mathcal{A}$  such that

$$(7.37) \quad F(p, t) = 1/p e^{pt} \int_0^t p e^{-pu} (H(p, u) - p \int_0^u H(p, v) dv) du$$

with the condition

$$(7.38) \quad \lim_{p \rightarrow \infty} p e^{-pt} F(p, t) \rightarrow 0 ,$$

which is equivalent to

$$(7.39) \quad H(p, t) = \mathbf{D}F(p, t) + e^{pt} \times \mathbf{D}F(p, t) .$$

Proof. If  $\mathbf{D}F(p, t) \equiv 0$  then, according to Theorem (6.29), there exists a function  $H(p, t) \in \mathcal{A}$  such that

$$(7.40) \quad \mathbf{D}F(p, t) = H(p, t) - p \int_0^t H(p, u) du .$$

Substituting this into (7.26), we obtain (7.37) with (7.38). Analogously to (4.5) and (4.4), the formula (7.40) is equivalent to (7.39). •

(7.41) Definition. The second D-derivative  $D^2F(p, t)$  of a function  $F(p, t) \in A$  is the D-derivative of  $DF(p, t)$ , that is,

$$D^2F(p, t) := D(DF(p, t)) .$$

In general, the  $k$ -th D-derivative  $D^kF(p, t)$  of a function  $F(p, t) \in A$  is the D-derivative of  $D^{k-1}F(p, t)$ , that is,

$$(7.42) \quad D^kF(p, t) := D(D^{k-1}F(p, t)) \quad \text{for } k = 1, 2, \dots$$

Here we put

$$D^0F(p, t) = F(p, t) ,$$

$$D^1F(p, t) = DF(p, t) . \bullet$$

(7.43) Theorem. For every function  $F(p, t) \in A$  we have

$$D^k(D^lF(p, t)) = D^{k+l}F(p, t) .$$

Proof. The theorem follows immediately from Definition (7.41). •

(7.44) Theorem. For every function  $F(p, t) \in A$  and  $k = 0, 1, 2, \dots$  we have

$$(7.45) \quad D^kF(p, t) = \sum_{j=0}^k (-1)^{j+1} \binom{k}{j} (j/p)^{j-1} (t^j - e^j) \frac{d^{k-j}}{dp^{k-j}} F(p, t) .$$

Proof. We shall prove the theorem by induction. By virtue of Definition (7.1), the formula (7.45) is true for  $k = 1$ . Thus, let us assume that it is true for  $k = r$ , that is,

$$D^rF(p, t) = \sum_{j=0}^r (-1)^{j+1} \binom{r}{j} (j/p)^{j-1} (t^j - e^j) \frac{d^{r-j}}{dp^{r-j}} F(p, t) .$$

Then we have

$$D^{r+1}F(p, t) = D(D^rF(p, t)) = \frac{d}{dp} D^rF(p, t) + (1/p - t) D^rF(p, t) =$$

$$\begin{aligned}
 &= \sum_{j=0}^r (-1)^{j+1} \binom{r}{j} (-j/p^2 - \ell^{j-1}) s^{-j/p^r - j} F(p, t) + \\
 &+ \sum_{j=0}^r (-1)^{j+1} \binom{r}{j} (j/p - \ell^{j-1} - \ell^j) s^{+t-j/p^r + t-j} F(p, t) + \\
 &+ \sum_{j=0}^r (-1)^{j+1} \binom{r}{j} (j/p^2 - \ell^{j-1} - 1/p \ell^j - j/p \ell^j + \ell^{j+1}) s^{-j/p^r - j} F(p, t) = \\
 &= \sum_{j=0}^r (-1)^{j+1} \binom{r}{j} (j/p - \ell^{j-1} - \ell^j) s^{+t-j/p^r + t-j} F(p, t) - \\
 &- \sum_{j=0}^r (-1)^{j+1} \binom{r}{j} ((r+1)/p \ell^j - \ell^{j+1}) s^{-j/p^r - j} F(p, t) = \\
 &= \sum_{j=0}^r (-1)^{j+1} \binom{r}{j} (j/p - \ell^{j-1} - \ell^j) s^{+t-j/p^r + t-j} F(p, t) - \\
 &- \sum_{j=1}^{r+1} (-1)^j \binom{r}{j-1} (j/p - \ell^{j-1} - \ell^j) s^{+t-j/p^r + t-j} F(p, t) = \\
 &= s^{+t/p^r + t} F(p, t) + \\
 &+ \sum_{j=1}^{r+1} (-1)^{j+1} \binom{r+1}{j} (j/p - \ell^{j-1} - \ell^j) s^{+t-j/p^r + t-j} F(p, t) + \\
 &+ (-1)^{r+2} ((r+1)/p \ell^r - \ell^{r+1}) F(p, t) = \\
 &= \sum_{j=0}^{r+1} (-1)^{j+1} \binom{r+1}{j} (j/p - \ell^{j-1} - \ell^j) s^{+t-j/p^r + t-j} F(p, t) .
 \end{aligned}$$

We see that if (7.45) is true for  $k=r$  then it is true also for  $k=r+1$ . This completes the proof. •

(7.46) Theorem. If  $g(p) \in A$  then we have for  $k = 0, 1, 2, \dots$

$$(7.47) \quad D^k g(p) \equiv d^k/dp^k g(p)$$

Proof. We shall prove the theorem by induction. Since (7.47) is true for  $k = 0$ , let us suppose that it is true for  $k = r$ , that is,

$$D^r g(p) \equiv d^r/dp^r g(p).$$

Then we have, in view of Theorem (7.10),

$$\begin{aligned} D^{r+1} g(p) &= D(D^r g(p)) \equiv D(d^r/dp^r g(p)) = \\ &= d^{r+1}/dp^{r+1} g(p) + (1/p - t) d^r/dp^r g(p). \end{aligned}$$

Since it follows from (7.22) that

$$(1/p - t) d^r/dp^r g(p) \equiv 0,$$

we obtain

$$D^{r+1} g(p) \equiv d^{r+1}/dp^{r+1} g(p).$$

It means that if (7.47) is true for  $k = r$  then it is also true for  $k = r + 1$ . This completes the proof. •

(7.48) Theorem. If  $F(p, t), g(p) \in A$  and

$$(7.49) \quad F(p, t) \equiv g(p),$$

then we have for  $k = 0, 1, 2, \dots$

$$(7.50) \quad D^k F(p, t) \equiv d^k/dp^k g(p).$$

Proof. It follows from (7.49), by virtue of Theorem (7.10), that

$$D^k F(p, t) \equiv D^k g(p).$$

Hence, by Theorem (7.46), we obtain (7.50). •

(7.51) Theorem. If  $f(t) \in A$  then we have for  $k = 0, 1, 2, \dots$

$$(7.52) \quad D^k f(t) = (-1)^{k+1} (k/p) t^{k-1} - t^k f(t).$$

Proof. Substituting  $F(p, t) = f(t)$  in (7.45), we obtain (7.52). •

(7.53) Theorem. If  $f(t) F(p, t) \in A$  then we have for  $k = 0, 1, 2, \dots$

$$(7.54) \quad D^k (f(t) F(p, t)) = f(t) D^k F(p, t).$$

Proof. Replacing  $F(p, t)$  in (7.45) by  $f(t) F(p, t)$ , we obtain (7.54). •

(7.55) Examples. Let us find D-derivatives for some functions  $F(p, t) \in A$ .

1) For  $f(t) = t^r$ , where  $r$  is any real number, we have, according to the formula (7.52), for  $k = 0, 1, 2, \dots$

$$(7.56) \quad D^k t^r = (-1)^{k+1} (k/p) t^{r+k-1} - t^{r+k},$$

2) For  $g(p) = 1/p$  and  $k = 0, 1, 2, \dots$  we have

$$(7.57) \quad D^k 1/p = (-1)^k t^k / p,$$

because, according to (7.2), we have  $D 1/p = -t/p$ , and supposing that (7.57) is true for  $k = r$ , we have by Theorem (7.53)

$$D^{r+1} 1/p = D(D^r 1/p) = D((-1)^r t^r / p) = (-1)^r t^r D 1/p = (-1)^{r+1} t^{r+1} / p,$$

that is, (7.57) for  $k = r + 1$ .

3) For  $g(p) = e^{rp}$ , where  $r$  is any real number, and for  $k = 0, 1, 2, \dots$  we have

$$(7.58) \quad D^k e^{rp} = (k/p) (r-t)^{k-1} + (r-t)^k e^{rp}.$$

We prove this formula by induction. It is obvious that (7.58) is true for  $k = 0$ . Therefore, supposing that it is true for  $k = q$ , we have

$$\begin{aligned} D^{q+1} e^{rp} &= D(D^q e^{rp}) = D\left(\left(q/p(r-t)R^{-1} + (r-t)R\right)e^{rp}\right) = \\ &= -q/p^2(r-t)R^{-1}e^{rp} + rq/p(r-t)R^{-1}e^{rp} + r(r-t)R e^{rp} + \\ &\quad + q/p^2(r-t)R^{-1}e^{rp} + 1/p(r-t)R e^{rp} - q/p t(r-t)R^{-1}e^{rp} - \\ &\quad - t(r-t)R e^{rp} = \\ &= ((q+1)/p(r-t)R + (r-t)R + 1)e^{rp}, \end{aligned}$$

that is, (7.58) for  $k = q+1$ . This completes the proof of the formula (7.58). •

(7.59) Theorem. If

$$(7.60) \quad g(p) \equiv 0, \quad g(p) \in R,$$

then for any real positive integer  $n$

$$(7.61) \quad d^n g(p)/dp^n \equiv 0, \quad d^n g(p)/dp^n \in R.$$

Proof. It follows from Definition (6.1) that  $d^n g(p)/dp^n \in R$ , because  $d^p(d^n g(p)/dp^n)/dp^p = d^{p+n}g(p)/dp^{p+n}$ . Moreover, according to Theorem (7.8), we have  $D^n g(p) \equiv 0$  and, by Theorem (7.16),  $d^n g(p)/dp^n \equiv 0$ . •

(7.62) Theorem. We have

$$(7.63) \quad g(p) \equiv 0, \quad g(p) \in R,$$

iff

$$(7.64) \quad \int_p^\infty g(u) du \equiv 0, \quad \int_p^\infty g(u) du \in R.$$

Proof. Since

$$\frac{d}{dp} \int_p^{\infty} g(u) du = -g(p),$$

it follows from Theorem (7.59) that (7.64) implies (7.63). Thus, in view of Theorem (6.76), it is sufficient to prove that

$$e^{pt} g(p) \in A \quad \text{implies} \quad e^{pt} \int_p^{\infty} g(u) du \in A.$$

Therefore, let us assume that

$$(7.65) \quad e^{pt} g(p) \in A.$$

According to (6.2), we have

$$|e^{pt} g(p)| \leq e^{kp} h_0(t),$$

that is,

$$|g(p)| \leq e^{-p(t+k)} h_0(t) \quad \text{for almost every } t \geq 0.$$

Replacing  $t$  with  $t+2k$ , we obtain

$$|g(p)| \leq e^{-p(t+2k)} h_0(t+2k) \quad \text{for almost every } t \geq 0.$$

Hence

$$\begin{aligned} \left| \int_p^{\infty} g(u) du \right| &\leq \int_p^{\infty} |g(u)| du \leq h_0(t+2k) \int_p^{\infty} e^{-u(t+2k)} du = \\ &= 1/(t+2k) h_0(t+2k) e^{-p(t+2k)} \end{aligned}$$

and

$$(7.66) \quad \left| e^{pt} \int_p^{\infty} g(u) du \right| \leq e^{-kp} H_0(t) \leq e^{kp} H_0(t),$$

where

$$H_0(t) := 1/(t+2k) h_0(t+2k).$$

According to (6.2), we have

$$(7.67) \quad \left| \frac{d^n}{dp^n} (e^{pt} g(p)) \right| \leq e^{kp} h_n(t) \quad \text{for } n = 0, 1, 2, \dots$$

We shall show that there exist analogously functions  $H_0(t), H_1(t), H_2(t), \dots$  such that

$$\left| \frac{d}{dp^n} \left( e^{pt} \int_p^\infty g(u) du \right) \right| \leq e^{pt} H_n(t) \quad \text{for } n = 0, 1, 2, \dots .$$

We have

$$\begin{aligned} \frac{d}{dp^n} \left( e^{pt} \int_p^\infty g(u) du \right) &= \sum_{j=0}^n \binom{n}{j} e^{-j} \frac{d}{dp^{n-j}} e^{pt} \frac{d}{dp^j} \int_p^\infty g(u) du = \\ &= e^{pt} \int_p^\infty g(u) du - s, \end{aligned}$$

where

$$\begin{aligned} s &:= \sum_{j=1}^n \binom{n}{j} e^{-j} \frac{d}{dp^{n-j}} e^{pt} \frac{d^{j-1}}{dp^{j-1}} g(p) = \\ &= \sum_{s=0}^{n-1} \binom{n+s}{s} e^{-s-1} \frac{d}{dp^{n-s-1}} e^{pt} \frac{d^s}{dp^s} g(p) = \\ &= \sum_{s=0}^{n-1} \sum_{j=s}^{n-1} \binom{n}{s} \binom{j}{s} e^{-s-1} \frac{d}{dp^{n-s-1}} e^{pt} \frac{d^s}{dp^s} g(p) = \\ &= \sum_{j=0}^{n-1} \sum_{s=0}^j \binom{n}{s} \binom{j}{s} e^{-s} \frac{d}{dp^{j-s}} e^{pt} \frac{d^s}{dp^s} g(p) = \\ &= \sum_{j=0}^{n-1} (e^{-j-1} \frac{d}{dp^j} (e^{pt} g(p))) . \end{aligned}$$

Hence, by virtue of (7.66) and (7.67), we obtain

$$\begin{aligned} \left| \frac{d}{dp^n} \left( e^{pt} \int_p^\infty g(u) du \right) \right| &\leq e^{pt} e^{dp} H_0(t) + \sum_{j=0}^{n-1} (e^{-j-1} \frac{d}{dp^j} h_j(t)) = \\ &= e^{dp} H_n(t) \quad \text{for } n = 0, 1, 2, \dots , \end{aligned}$$

where

$$H_p(t) := t^p H_0(t) + \sum_{j=0}^{p-1} t^{p-j-1} h_j(t).$$

This means that the function

$$\rho^t \int_0^\infty g(u) du$$

satisfies the condition (6.2) from Definition (6.1). Since it satisfies also the remaining conditions from that definition, we obtain

$$(7.68) \quad \rho^t \int_0^\infty g(u) du \in R,$$

which means that (7.65) implies (7.68). This completes the proof. •

## VIII. Further applications

In this chapter we shall give some examples of new techniques, using functions of two variables  $F(p, t)$ . Calculating with such functions, we must remember that, generally, we cannot multiply in the ordinary sense any equivalence on both sides by a function  $F(p, t)$ , in particular by a function  $f(t)$ , but we may do it in the sense of convolution, according to (4.19). We may also multiply in the ordinary sense both sides of an equivalence by a function  $g(p)$ , according to (4.27).

At the end of this chapter we shall find an example of a problem, which cannot be solved by the Laplace transformation, and cannot be solved by use of Mikusinski's theory, but will be solved by methods of the presented theory.

First, we shall work out some useful formulas.

(8.1) Theorem. We have

$$(8.2) \quad \sum_{j=0}^n (-1)^j \binom{k}{j} = (-1)^n \binom{k-1}{n} \quad \text{for } 0 \leq n \leq k-1,$$

$$(8.3) \quad \sum_{j=0}^n (-1)^j \binom{n}{j} = \begin{cases} 1 & \text{for } n=0, \\ 0 & \text{for } n>0, \end{cases}$$

$$(8.4) \quad \sum_{j=0}^n (-1)^j \binom{k}{j} \binom{n+j}{n} = (-1)^n \binom{k-1}{n} \quad \text{for } n \leq k-1, n \leq k-n-1.$$

Proof. The formula (8.2) is true for  $n=0$ . Supposing that it is true for an  $n \leq k-2$ , we obtain

$$\begin{aligned}
 \sum_{j=0}^{n+1} (-1)^j \binom{k}{j} &= \sum_{j=0}^n (-1)^j \binom{k}{j} + (-1)^{n+1} \binom{k}{n+1} = \\
 &= (-1)^n \binom{k-1}{n} + (-1)^{n+1} \binom{k}{n+1} = \\
 &= (-1)^{n+1} \left( \binom{k}{n+1} - \binom{k-1}{n} \right) = \\
 &= (-1)^{n+1} \binom{k-1}{n+1}.
 \end{aligned}$$

By induction, the formula (8.2) is true for  $n = 0, 1, \dots, k-1$ .

The sum in (8.3) is Newton's expansion of  $(x-1)^n$ . Hence we obtain the formula (8.3).

It follows from (8.2) that the formula (8.4) is true for  $a=0$ . Since (8.4) is also true for  $a=0$ , it is sufficient to prove, that if it is true for  $a=u+r$  with  $n=r$  and for  $a=u$  with  $n=r+1$  then it is also true for  $a=u+r$  with  $n=r+1$ , where  $u \leq k-2$  and  $r \leq k-u-3$ .

Thus let us suppose that

$$\sum_{j=0}^r (-1)^j \binom{k}{j} \binom{u+r+t-j}{u+1} = (-1)^r \binom{k-u-2}{r}$$

and

$$\sum_{j=0}^{r+1} (-1)^j \binom{k}{j} \binom{u+r+t-j}{u} = (-1)^{r+1} \binom{k-u-1}{r+1}.$$

Then we have

$$\begin{aligned}
 & \sum_{j=0}^{r+1} (-1)^j \binom{k}{j} \binom{u+r+2-j}{u+1} = \\
 & = \sum_{j=0}^{r+1} (-1)^j \binom{k}{j} \left( \binom{u+r+1-j}{u+1} + \binom{u+r+1-j}{u} \right) = \\
 & = \sum_{j=0}^r (-1)^j \binom{k}{j} \binom{u+r+1-j}{u+1} + \sum_{j=0}^{r+1} (-1)^j \binom{k}{j} \binom{u+r+1-j}{u} = \\
 & = (-1)^r \binom{k-u-2}{r} + (-1)^{r+1} \binom{k-u-1}{r+1} = (-1)^{r+1} \binom{k-u-2}{r+1}
 \end{aligned}$$

which means that the formula (8.4) is also true for  $a = u+1$  with  $n = r+1$ . This completes the proof. •

(8.5) Theorem. If functions  $F(p, t)$  and  $x(p, t)$  have derivatives with respect to  $t$  up to  $\frac{\partial F(p, t)}{\partial t^k}$  and  $\frac{\partial x(p, t)}{\partial t^k}$ , respectively, then

$$(8.6) \quad F(p, t) \frac{\partial x(p, t)}{\partial t^k} = \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{\partial^j (F(p, t)/\partial t^j) x(p, t)}{\partial t^{k-j}} .$$

Proof. Introducing, for simplicity,

$$F := F(p, t) = F^{(0)}, \quad x := x(p, t) = x^{(0)},$$

and

$$F^{(j)} := \frac{\partial F(p, t)}{\partial t^j}, \quad x^{(j)} := \frac{\partial x(p, t)}{\partial t^j} \quad \text{for } j = 0, 1, \dots, k ,$$

we have

$$(Fx)^{(k)} = \sum_{j=0}^k \binom{k}{j} F^{(j)} x^{(k-j)} .$$

Hence

$$\binom{k}{0} f^{(0)} x^{(k)} + \binom{k}{1} f^{(1)} x^{(k-1)} + \binom{k}{2} f^{(2)} x^{(k-2)} + \dots + \binom{k}{k} f^{(k)} x^{(0)} = (f^{(0)} x)^{(k)}$$

$$\binom{k-1}{0} f^{(1)} x^{(k-1)} + \binom{k-1}{1} f^{(2)} x^{(k-2)} + \dots + \binom{k-1}{k-1} f^{(k)} x^{(0)} = (f^{(1)} x)^{(k-1)}$$

$$\binom{k-2}{0} f^{(2)} x^{(k-2)} + \dots + \binom{k-2}{k-2} f^{(k)} x^{(0)} = (f^{(2)} x)^{(k-2)}$$

.....

$$\binom{0}{0} f^{(k)} x^{(0)} = (f^{(k)} x)^{(0)}$$

Solving the above system of equations with respect to  $f^{(0)} x^{(k)}$  we obtain

$$(8.7) \quad f^{(0)} x^{(k)} = \begin{vmatrix} (f^{(0)} x)^{(k)} & \binom{k}{1} & \binom{k}{2} & \dots & \binom{k}{k} \\ (f^{(1)} x)^{(k-1)} & \binom{k-1}{0} & \binom{k-1}{1} & \dots & \binom{k-1}{k-1} \\ (f^{(2)} x)^{(k-2)} & \binom{k-2}{0} & \dots & \binom{k-2}{k-2} \\ \dots & & & & \\ (f^{(k)} x)^{(0)} & \binom{0}{0} & & & \end{vmatrix} =$$

$$= \sum_{j=0}^k (-1)^j (f^{(j)} x)^{(k-j)} \Delta_{kj},$$

where

$$\Delta_{kj} := \begin{vmatrix} \binom{k}{1} & \binom{k}{2} & \dots & \binom{k}{j-1} & \binom{k}{j} \\ \binom{k-1}{0} & \binom{k-1}{1} & \dots & \binom{k-1}{j-2} & \binom{k-1}{j-1} \\ \binom{k-2}{0} & \dots & \binom{k-2}{j-3} & \binom{k-2}{j-2} \\ \dots & & \dots & & \dots \\ \binom{k-j+1}{0} & \binom{k-j+1}{1} \end{vmatrix} =$$

$$= \frac{\frac{t}{1!} \frac{t}{2!} \frac{t}{3!} \frac{t}{4!} \dots \frac{t}{(j-1)!} \frac{t}{j!}}{\frac{t}{0!} \frac{t}{1!} \frac{t}{2!} \frac{t}{3!} \dots \frac{t}{(j-2)!} \frac{t}{(j-1)!}} =$$

$$= \frac{k! (k-1)! \dots (k-j+1)!}{(k-1)! (k-2)! \dots (k-j)!} =$$

$$= \frac{1}{0!} \frac{1}{1!} \frac{1}{2!} \dots \frac{1}{(j-3)!} \frac{1}{(j-2)!} =$$

$$= \frac{t}{0!} \frac{t}{1!}$$

$$= \frac{k!}{(k-j)!} \frac{0!1! \dots (j-1)!}{1!2! \dots j!} \begin{vmatrix} (0) & (0) & (0) & (0) & \dots & (0) & (0) \\ (1) & (1) & (1) & (1) & \dots & (1) & (1) \\ (2) & (2) & (2) & (2) & \dots & (2) & (2) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ (j-v) & (j-v) & (j-v) & (j-v) & \dots & (j-v) & (j-v) \end{vmatrix}$$

Subtracting from the first row of the above determinant all even rows and adding the remaining odd rows, we obtain in the first  $j-1$  columns of the first row Newton's expansions of  $(1-x)^q$  for  $q = 1, \dots, j-1$  and in the last column the expansion of  $(1-x)^{j-1} + (1-x)^{j+1}$ . Hence we obtain

$$\Delta_{kj} = \frac{k!}{(k-j)!j!} \begin{vmatrix} 0 & 0 & 0 & 0 & \dots & 0 & (1-x)^{j+1} \\ (1) & (1) & (1) & (1) & \dots & (1) & (1) \\ (2) & (2) & (2) & (2) & \dots & (2) & (2) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ (j-v) & (j-v) & (j-v) & (j-v) & \dots & (j-v) & (j-v) \end{vmatrix} = (j)$$

Substituting this in (8.7), we obtain (8.6). •

(8.8) Theorem. If  $F(p, t)$ ,  $dF(p, t)/dt$ , ...,  $d^k F(p, t)/dt^k \in F$  and  $x(p, t)$ ,  $dx(p, t)/dt$ , ...,  $d^k x(p, t)/dt^k \in F$ , where  $k$  is a non-negative integer, then

$$(8.9) \quad F(p, t) d^k x(p, t)/dt^k = x(p, t) \sum_{j=0}^k (-1)^j \binom{k}{j} p^{k-j} d^j F(p, t)/dt^j - \\ - \sum_{n=0}^{k-1} p^{k-n} \sum_{m=0}^n (-1)^{m-n} \binom{k-n-1}{m-n} [d^{m-n} F(p, t)/dt^{m-n} dx(p, t)/dt^m]_{t=0}$$

which is equivalent to

$$(8.10) \quad F(p, t) d^k x(p, t)/dt^k = x(p, t) \sum_{j=0}^k (-1)^j \binom{k}{j} p^{k-j} d^j F(p, t)/dt^j - \\ - \sum_{n=0}^{k-1} [dx(p, t)/dt^n]_{t=0} \sum_{m=n}^{k-1} (-1)^{m-n} \binom{k-n-1}{m-n} p^{k-m} [d^{m-n} F(p, t)/dt^{m-n}]_{t=0},$$

and also equivalent to

$$(8.11) \quad F(p, t) d^k x(p, t)/dt^k = x(p, t) \sum_{j=0}^k (-1)^j \binom{k}{j} p^{k-j} d^j F(p, t)/dt^j - \\ - \sum_{n=0}^{k-1} [d^j F(p, t)/dt^n]_{t=0} \sum_{m=0}^{k-n-1} (-1)^{m-n} \binom{k-n-1}{m-n} p^{k-m-n} [dx(p, t)/dt^n]_{t=0}.$$

Proof. According to (4.79), we obtain from (8.6)

$$F(p, t) d^k x(p, t)/dt^k = \\ = \sum_{j=0}^k (-1)^j \binom{k}{j} (p^{k-j} d^j F(p, t)/dt^j x(p, t)) - \sum_{r=0}^{k-j-1} p^{k-j-r} [x(d^r F(p, t)/dt^r x(p, t))/dt^r]_{t=0}$$

which is equivalent to

$$(8.12) \quad F(p, t) d^k x(p, t)/dt^k = x(p, t) \sum_{j=0}^k (-1)^j \binom{k}{j} p^{k-j} d^j F(p, t)/dt^j - s,$$

where

$$\begin{aligned}
 s &:= \sum_{j=0}^k (-1)^j \binom{k}{j} \sum_{r=0}^{k-j-1} p^{k-j-r} \left[ \partial^r \left( \partial^j F(p, t) / \partial t^j \right) X(p, t) \right]_{t=0} = \\
 &= \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \sum_{r=0}^{k-j-1} \sum_{s=0}^r \binom{r}{s} \left[ \partial^{j+s} F(p, t) / \partial t^{j+s} \partial^{r-s} X(p, t) / \partial t^{r-s} \right]_{t=0} = \\
 &= \sum_{j=0}^{k-1} \sum_{r=0}^{k-j-1} \sum_{s=0}^r (-1)^j \binom{k}{j} \binom{r}{s} p^{k-j-r} \left[ \partial^{j+s} F(p, t) / \partial t^{j+s} \partial^{r-s} X(p, t) / \partial t^{r-s} \right]_{t=0}.
 \end{aligned}$$

Introducing

$$m := j + r , \quad n := r - s ,$$

we have

$$s = m - n - j , \quad r = m - j , \quad j + s = m - n$$

and

$$\begin{aligned}
 s &= \sum_{j=0}^{k-1} \sum_{m=j}^{k-1} \sum_{n=0}^{m-j} (-1)^j \binom{k}{j} \binom{m}{n} p^{k-m} \left[ \partial^{m-n} F(p, t) / \partial t^{m-n} \partial^n X(p, t) / \partial t^n \right]_{t=0} = \\
 &= \sum_{m=0}^{k-1} p^{k-m} \sum_{n=0}^m \left[ \partial^{m-n} F(p, t) / \partial t^{m-n} \partial^n X(p, t) / \partial t^n \right]_{t=0} \sum_{j=0}^{m-n} (-1)^j \binom{k}{j} \binom{m}{n} = \\
 &= \sum_{n=0}^{k-1} \left[ \partial^n X(p, t) / \partial t^n \right]_{t=0} \sum_{m=n}^{k-1} p^{k-m} \left[ \partial^{m-n} F(p, t) / \partial t^{m-n} \right]_{t=0} \sum_{j=0}^{m-n} (-1)^j \binom{k}{j} \binom{m}{n}
 \end{aligned}$$

and, according to the formula (8.4),

$$\begin{aligned}
 s &= \sum_{m=0}^{k-1} p^{k-m} \sum_{n=0}^m (-1)^{m-n} \binom{k-n-1}{m-n} \left[ \partial^{m-n} F(p, t) / \partial t^{m-n} \partial^n X(p, t) / \partial t^n \right]_{t=0} = \\
 &= \sum_{n=0}^{k-1} \left[ \partial^n X(p, t) / \partial t^n \right]_{t=0} \sum_{m=n}^{k-1} (-1)^{m-n} \binom{k-n-1}{m-n} p^{k-m} \left[ \partial^{m-n} F(p, t) / \partial t^{m-n} \right]_{t=0} .
 \end{aligned}$$

Substituting this into (8.12), we obtain (8.9) and (8.10). Introducing

$$r := s - n ,$$

we obtain

$$\begin{aligned} s &= \sum_{n=0}^{k-1} [\partial^n x(p, t)/\partial t^n]_{t=0} \sum_{r=0}^{k-n-1} (-1)^r \binom{k-n-1}{r} p^{k-r-n} [\partial^r F(p, t)/\partial t^r]_{t=0} = \\ &= \sum_{r=0}^{k-1} [\partial^r F(p, t)/\partial t^r]_{t=0} \sum_{n=0}^{k-r-1} (-1)^n \binom{k-n-1}{r} p^{k-r-n} [\partial^n x(p, t)/\partial t^n]_{t=0} . \end{aligned}$$

Replacing  $r$  by  $s$  and substituting  $s$  into (8.12), we obtain (8.11). •

(8.13) Theorem. If  $x(p, t)$ ,  $\partial x(p, t)/\partial t$ , ...,  $\partial^k x(p, t)/\partial t^k \in F$ , where  $k$  is a non-negative integer, then for any non-negative integer  $q$  we have

$$(8.14) \quad \frac{\partial^q}{q!} \frac{\partial^k x(p, t)/\partial t^k}{q!} = \left\{ \begin{array}{l} x(p, t) \sum_{j=0}^k (-1)^j \binom{k}{j} p^{k-j} t^{q-j}/(q-j)! \quad \text{for } q \geq k , \\ x(p, t) \sum_{j=0}^q (-1)^j \binom{k}{j} p^{k-j} t^{q-j}/(q-j)! - \\ - (-1)^q \sum_{n=0}^{k-q-1} \binom{k-n-1}{q} p^{k-q-n} [\partial^n x(p, t)/\partial t^n]_{t=0} \quad \text{for } q < k . \end{array} \right.$$

Proof. We obtain (8.14) when substituting  $F(p, t) := \frac{\partial^k}{k!} x(p, t)$  in (8.11). •

(8.15) Theorem. If  $x(p, t), dx(p, t)/dt, \dots, d^k x(p, t)/dt^k \in F$ , where  $k$  is a non-negative integer, then

$$(8.16) \quad t d^k x(p, t)/dt^k = (p^k t - k p^{k-1}) x(p, t) + \sum_{j=0}^{k-2} (k-j-1) p^{k-j-1} [dx(p, t)/dt]_{t=0} .$$

In particular, we have

$$(8.17) \quad t dx(p, t)/dt = (pt - 1) x(p, t) ,$$

$$(8.18) \quad t d^2 x(p, t)/dt^2 = (p^2 t - 2p) x(p, t) + p x(p, 0) ,$$

$$(8.19) \quad t d^3 x(p, t)/dt^3 = (p^3 t - 3p^2) x(p, t) + (2p^2 x(p, 0) + p [dx(p, t)/dt]_{t=0}) .$$

Proof. The formula (8.16) is equivalent to (8.14) with  $q = 1$ . We obtain the formulas (8.17), (8.18) and (8.19) from (8.16), when putting  $k = 1, 2, 3$ . •

(8.20) Theorem. If  $x(p, t), dx(p, t)/dt, \dots, d^k x(p, t)/dt^k \in F$ , where  $k$  is a non-negative integer, then for any non-negative integer  $q$  we have

$$(8.21) \quad e^{qt} d^k x(p, t)/dt^k = (p-q)^k e^{qt} x(p, t) - p \sum_{j=0}^{k-1} (p-q)^{k-j-1} [dx(p, t)/dt]_{t=0} .$$

In particular, we have

$$(8.22) \quad e^{qt} dx(p, t)/dt = (p-q) e^{qt} x(p, t) - p x(p, 0) ,$$

$$(8.23) \quad e^{qt} d^2 x(p, t)/dt^2 = (p-q)^2 e^{qt} x(p, t) - p(p-q) x(p, 0) - p [dx(p, t)/dt]_{t=0} ,$$

$$(8.24) \quad e^{qt} d^3 x(p, t)/dt^3 = (p-q)^3 e^{qt} x(p, t) - p(p-q)^2 x(p, 0) - p(p-q) [dx(p, t)/dt]_{t=0} - p [d^2 x(p, t)/dt^2]_{t=0} .$$

Proof. Substituting  $F(p, t) := e^{qt}$  in (8.10), we have

$$(8.25) \quad e^{qt} \frac{d^k x(p, t)/dt^k}{dt^k} = x(p, t) \sum_{j=0}^k (-1)^j \binom{k}{j} p^{k-j} q^j e^{qt} - s = \\ = (p-q, k) e^{qt} x(p, t) - s,$$

where

$$s = \sum_{j=0}^{k-1} [\frac{dx(p, t)/dt^j}{dt^j}]_{t=0} \sum_{n=j}^{k-1} (-1)^{n-j} \binom{k-j-1}{n-j} p^{k-n} q^{n-j} = \\ = p \sum_{j=0}^{k-1} [\frac{dx(p, t)/dt^j}{dt^j}]_{t=0} \sum_{n=0}^{k-j-1} \binom{k-j-1}{n} p^{k-j-1-n} (-q)^n = \\ = p \sum_{j=0}^{k-1} (p-q, k-j-1) [\frac{dx(p, t)/dt^j}{dt^j}]_{t=0}.$$

Substituting it into (8.25), we obtain (8.21). •

(8.26) Theorem. If  $F(p, t)$ ,  $dF(p, t)/dt$ , ...,  $d^k F(p, t)/dt^k \in F$  and  $x(p, t)$ ,  $dx(p, t)/dt$ , ...,  $d^k x(p, t)/dt^k \in F$ , where  $k$  is a non-negative integer, then

$$(8.27) \quad F(p, t) \frac{d^k x(p, t)/dt^k}{dt^k} \equiv x(p, t) \sum_{j=0}^k (-1)^j \binom{k}{j} p^{k-j} dF(p, t)/dt^j - \\ - \sum_{n=0}^{k-1} p^{k-n} \sum_{m=0}^n (-1)^{n-m} \binom{k-n-1}{n-m} [d^{n-m} F(p, t)/dt^{n-m} dx(p, t)/dt^m]_{t=0}$$

which is equivalent to

$$(8.28) \quad F(p, t) \frac{d^k x(p, t)/dt^k}{dt^k} \equiv x(p, t) \sum_{j=0}^k (-1)^j \binom{k}{j} p^{k-j} dF(p, t)/dt^j - \\ - \sum_{n=0}^{k-1} [dx(p, t)/dt^n]_{t=0} \sum_{m=n}^{k-1} (-1)^{m-n} \binom{k-n-1}{m-n} p^{k-m} [d^{m-n} F(p, t)/dt^{m-n}]_{t=0}$$

and equivalent to

$$(8.29) \quad F(p, t) \frac{d^k x(p, t)/dt^k}{t=0} \equiv x(p, t) \sum_{j=0}^k (-1)^j \binom{k}{j} p^{k-j} \frac{d^j F(p, t)/dt^j}{t=0} - \\ - \sum_{n=0}^{k-1} \left[ \frac{d^n F(p, t)/dt^n}{t=0} \right] \sum_{n=0}^{k-n-1} (-1)^n \binom{k-n-1}{n} p^{k-n} \left[ \frac{d^n x(p, t)/dt^n}{t=0} \right].$$

Proof. We prove the theorem analogously to Theorem (8.8), using (6.66) instead of (4.79). •

(8.30) Theorem. If  $x(p, t)$ ,  $dx(p, t)/dt$ , ...,  $d^k x(p, t)/dt^k \in A$ , where  $k$  is a non-negative integer, then for any non-negative integer  $q$  we have

$$(8.31) \quad \frac{d^q}{dt^q} \frac{d^k x(p, t)/dt^k}{t=0} \equiv \begin{cases} x(p, t) \sum_{j=0}^k (-1)^j \binom{k}{j} p^{k-j} t^{j/(q-j)} & \text{for } q \geq k, \\ x(p, t) \sum_{j=0}^q (-1)^j \binom{k}{j} p^{k-j} t^{j/(q-j)} - \\ - (-1)^q \sum_{n=0}^{k-q-1} \binom{k-n-1}{q} p^{k-q-n} \left[ \frac{d^n x(p, t)/dt^n}{t=0} \right] & \text{for } q < k. \end{cases}$$

Proof. We obtain (8.31) when substituting  $F(p, t) := \frac{d^q}{dt^q}$  in (8.29). •

(8.32) Theorem. If  $x(p, t)$ ,  $dx(p, t)/dt$ , ...,  $d^k x(p, t)/dt^k \in A$ , where  $k$  is a non-negative integer, then

$$(8.33) \quad t \frac{d^k x(p, t)/dt^k}{t=0} \equiv (p^k t - k p^{k-1}) x(p, t) + \sum_{j=0}^{k-2} (k-j-1) p^{k-j-1} \left[ \frac{d^j x(p, t)/dt^j}{t=0} \right].$$

In particular, we have

$$(8.34) \quad t \frac{dx(p, t)}{dt} \equiv (pt - 1) x(p, t),$$

$$(8.35) \quad t \frac{d^2x(p, t)}{dt^2} \equiv (p^2t - 2p) x(p, t) + p x(p, 0),$$

$$(8.36) \quad t \frac{d^3x(p, t)}{dt^3} \equiv (p^3t - 3p^2) x(p, t) + (2p^2 x(p, 0) + p \{ \frac{dx(p, t)}{dt} \}_{t=0}).$$

Proof. The formula (8.33) is equivalent to (8.31) with  $q = r$ . We obtain the formulas (8.34), (8.35) and (8.36) from (8.33), when putting  $k = 1, 2, 3$ . •

(8.37) Theorem. If  $x(t)$ ,  $dx(t)/dt$ , ...,  $d^k x(t)/dt^k \in \mathbb{A}$ , where  $k$  is a non-negative integer, and there exists a function  $x(p) \in \mathbb{A}$  such that

$$(8.38) \quad x(t) \equiv x(p)$$

then for any non-negative integer  $q$  we have

$$(8.39) \quad \frac{d^q}{dt^q} x^{(k)}(t) \equiv \begin{cases} \sum_{j=0}^q (-1)^j \frac{1/p+j-1}{j!} \frac{d^j x(p)}{dp^j} & \text{for } k=0, \\ (-1)^q \sum_{j=q-k+1}^q \binom{k-1}{q-j} p^{j-q+k-1} \frac{1}{j!} \frac{d^j x(p)}{dp^j} & \text{for } q \geq k \geq 1, \\ (-1)^q \sum_{j=0}^q \binom{k-1}{q-j} p^{k-q+j-1} \frac{1}{j!} \frac{d^j x(p)}{dp^j} - \\ - (-1)^q \sum_{j=0}^{k-q-1} \binom{k-j-1}{q} p^{k-q-j} x^{(j)}(0) & \text{for } q < k. \end{cases}$$

In particular, we have

$$(8.40) \quad t^q/q! \ x(t) \equiv \sum_{j=0}^q (-1)^j \ 1/p^{q-j} \ 1/j! \ \frac{d^j X(p)}{dp^j}$$

$$(8.41) \quad t x^{(k)}(t) \equiv -(k-1) p^{k-1} X(p) - p^k \frac{dX(p)}{dp} + \sum_{j=0}^{k-2} (k-j-1) p^{k-j-1} x^{(j)}(0)$$

$$(8.42) \quad t^k/k! \ x^{(k)}(t) \equiv (-1)^k \sum_{j=1}^k (-1)^j p^j \ 1/j! \ \frac{d^j X(p)}{dp^j} \quad \text{for } k \geq 1$$

and

$$(8.43) \quad t \ x(t) \equiv 1/p X(p) - \frac{dX(p)}{dp}$$

$$(8.44) \quad t^2 \ x(t) \equiv 2/p^2 X(p) - 2/p \frac{dX(p)}{dp} + \frac{d^2 X(p)}{dp^2}$$

$$(8.45) \quad t^3 \ x(t) \equiv 6/p^3 X(p) - 6/p^2 \frac{dX(p)}{dp} + 3/p \frac{d^2 X(p)}{dp^2} - \frac{d^3 X(p)}{dp^3}$$

$$(8.46) \quad t \ x'(t) \equiv -p \frac{dX(p)}{dp}$$

$$(8.47) \quad t^2 \ x''(t) \equiv p \frac{d^2 X(p)}{dp^2}$$

$$(8.48) \quad t^3 \ x''(t) \equiv -p \frac{d^3 X(p)}{dp^3}$$

$$(8.49) \quad t \ x'''(t) \equiv -p (X(p) - x(0)) - p^2 \frac{dX(p)}{dp}$$

$$(8.50) \quad t^2 \ x'''(t) \equiv 2p \frac{dX(p)}{dp} + p^2 \frac{d^2 X(p)}{dp^2}$$

$$(8.51) \quad t^3 \ x'''(t) \equiv -3p \frac{d^2 X(p)}{dp^2} - p^3 \frac{d^3 X(p)}{dp^3}$$

$$(8.52) \quad t \ x''''(t) \equiv p x'(0) - 2p^2 (X(p) - x(0)) - p^3 \frac{dX(p)}{dp}$$

$$(8.53) \quad t^2 \ x''''(t) \equiv 2p(X(p) - x(0)) + 4p^2 \frac{dX(p)}{dp} + p^3 \frac{d^2 X(p)}{dp^2}$$

$$(8.54) \quad t^3 \ x''''(t) \equiv -6p \frac{dX(p)}{dp} - 6p^2 \frac{d^2 X(p)}{dp^2} - p^3 \frac{d^3 X(p)}{dp^3}$$

Proof. First, we shall prove (8.40) by induction. By virtue of (8.38) the formula (8.40) is true for  $q=0$ . If (8.40) is true for any  $q$  then we have, according to (7.19)

$$\begin{aligned}
 & t^{q+1}/(q+1)! x(t) = t/(q+1) \cdot t^q/q! x(t) \equiv \\
 & \equiv t/p \cdot t/(q+1) \sum_{j=0}^q (-1)^j \frac{1}{p^j j!} \frac{1}{j!} \frac{d^j x(p)}{dp^j} + \\
 & + \frac{1}{(q+1)} \sum_{j=0}^q (-1)^j \frac{(q-j)}{p^j j!} \frac{1}{p^j j+1} \frac{1}{j!} \frac{d^j x(p)}{dp^j} - \\
 & - \frac{1}{(q+1)} \sum_{j=0}^q (-1)^j \frac{1}{p^j j!} \frac{1}{p^j j+1} \frac{d^{j+1} x(p)}{dp^{j+1}} = \\
 & = \frac{1}{(q+1)} \sum_{j=0}^q (-1)^j \frac{(q-j+1)}{p^j j!} \frac{1}{p^j j+1} \frac{1}{j!} \frac{d^j x(p)}{dp^j} + \\
 & + \frac{1}{(q+1)} \sum_{j=1}^{q+1} (-1)^j \frac{1}{p^j j+1} \frac{1}{j!} j \frac{d^j x(p)}{dp^j} = \\
 & = \frac{1}{p^{q+1}} x(p) + \sum_{j=1}^{q+1} (-1)^j \frac{1}{p^j j+1} \frac{1}{j!} \frac{d^j x(p)}{dp^j} + \\
 & + (-1)^{q+1} \frac{1}{(q+1)!} \frac{d^{q+1} x(p)}{dp^{q+1}} = \\
 & = \sum_{j=0}^{q+1} (-1)^j \frac{1}{p^j j+1} \frac{1}{j!} \frac{d^j x(p)}{dp^j} ,
 \end{aligned}$$

which means that the formula is then true also for  $n+1$ . This completes the proof of (8.40).

Now we shall prove the formula (8.39) by replacing  $x(t)$  in (8.40) by  $x^{(k)}(t)$  and (8.38) by

$$(8.55) \quad x^{(k)}(t) \equiv p^k x(p) - \sum_{s=0}^{k-1} p^{k-s} x^{(s)}(0) .$$

Thus we obtain

$$t^q/q! x^{(k)}(t) \equiv \sum_{j=0}^q (-1)^j \frac{1}{p^j j!} \left( p^k x(p) - \sum_{s=0}^{k-1} p^{k-s} x^{(s)}(0) \right) x_j .$$

Since this formula coincides for  $k=0$  with (8.39) and (8.40), let us assume that  $k \geq 1$ . Then we have

$$\begin{aligned}
 \frac{\partial^k}{q!} x^{(k)}(t) &= \sum_{j=0}^q (-1)^j \frac{1}{pt^j} \frac{1}{j!} (\partial^k x(p))_{j,j} - \\
 &- \frac{1}{pt^k} \sum_{s=0}^{k-1} p^{k-s} x^{(s)}(0) - \\
 &- \sum_{j=1}^q (-1)^j \frac{1}{pt^j} \frac{1}{j!} \sum_{s=0}^{k-1} (\partial^{k-s})_{j,j} x^{(s)}(0) = \\
 &= \sum_{j=0}^q (-1)^j \frac{1}{pt^j} \frac{1}{j!} \sum_{s=0}^j \binom{j}{s} (\partial^k)_{j,s} \partial^s x(p) / \partial p^s - \\
 &- \frac{1}{pt^k} \sum_{s=0}^{k-1} p^{k-s} x^{(s)}(0) - \\
 &- \sum_{j=1}^q (-1)^j \frac{1}{pt^j} \frac{1}{j!} \sum_{s=0}^{k-j} (\partial^{k-s})_{j,j} x^{(s)}(0) .
 \end{aligned}$$

Hence we obtain

$$\begin{aligned}
 (8.56) \quad \frac{\partial^k}{q!} x^{(k)}(t) &= \sum_{s=0}^q \frac{1}{s!} \partial^s x(p) / \partial p^s \sum_{j=s}^q (-1)^j \frac{1}{pt^j} \frac{1}{(j-s)!} (\partial^k)_{j,s} - \\
 &- \sum_{s=0}^{k-1} p^{k-s} x^{(s)}(0) - \\
 &- \sum_{j=1}^q (-1)^j \frac{1}{pt^j} \frac{1}{j!} \sum_{s=0}^{k-j} (\partial^{k-s})_{j,j} x^{(s)}(0) .
 \end{aligned}$$

From now on we shall consider separately the two following cases :  $q < k$  and  $q \geq k$ .

First let us suppose that  $q < k$ . Then we obtain from (8.56)

$$\begin{aligned}
 \frac{\partial}{\partial q!} x^{(k)}(t) &\equiv \sum_{s=0}^q p^{k-q+s} \frac{1}{s!} \frac{d^s X(p)}{dp^s} \sum_{j=s}^q (-1)^j \binom{k}{j-s} - \\
 &- \sum_{s=0}^{k-1} p^{k-q-s} x^{(s)}(0) - \\
 &- \sum_{j=1}^q \sum_{s=0}^{k-j} (-1)^j \binom{k-j}{j} p^{k-q-s} x^{(s)}(0) = \\
 &= \sum_{s=0}^q (-1)^s p^{k-q+s} \frac{1}{s!} \frac{d^s X(p)}{dp^s} \sum_{j=0}^{q-s} (-1)^j \binom{k}{j} - \\
 &- \sum_{s=0}^{k-1} p^{k-q-s} x^{(s)}(0) - \\
 &- \sum_{s=0}^{k-q} p^{k-q-s} x^{(s)}(0) \sum_{j=1}^q (-1)^j \binom{k-s}{j} - \\
 &- \sum_{s=k-q+1}^{k-1} p^{k-q-s} x^{(s)}(0) \sum_{j=1}^{k-s} (-1)^j \binom{k-s}{j}
 \end{aligned}$$

and, according to (8.2) and (8.3),

$$\begin{aligned}
 \frac{\partial}{\partial q!} x^{(k)}(t) &\equiv (-1)^q \sum_{s=0}^q \binom{k-1}{q-s} p^{k-q+s} \frac{1}{s!} \frac{d^s X(p)}{dp^s} - \\
 &- \sum_{s=0}^{k-1} p^{k-q-s} x^{(s)}(0) - \\
 &- \sum_{s=0}^{k-q} p^{k-q-s} x^{(s)}(0) \left( (-1)^q \binom{k-s-1}{q} - 1 \right) - \\
 &- \sum_{s=k-q+1}^{k-1} p^{k-q-s} x^{(s)}(0) \left( (1 - 1)^{k-s} - 1 \right) -
 \end{aligned}$$

$$= (-1)^q \sum_{s=0}^q \binom{k-1}{q-s} p^{k-q+s} \frac{1}{s!} \frac{d^s X(p)}{dp^s} -$$

$$- (-1)^q \sum_{s=0}^{k-q-1} \binom{k-s-1}{q} p^{k-q-s} x^{(s)}(0).$$

Replacing here  $s$  with  $j$  we obtain the formula (8.39) for  $q < k$ .

Now let us suppose that  $q \geq k+1$ . Since

$$\begin{aligned} & \sum_{j=1}^q (-1)^j \frac{1}{p^{q-j}} \frac{1}{j!} \sum_{s=0}^{k-j} \binom{k-s}{j} (f^{k-s})(j) x^{(s)}(0) = \\ &= \sum_{j=1}^k \sum_{s=0}^{k-j} (-1)^j \binom{k-s}{j} p^{k-q-s} x^{(s)}(0) = \\ &- \sum_{s=0}^{k-1} p^{k-q-s} x^{(s)}(0) \sum_{j=1}^{k-s} (-1)^j \binom{k-s}{j} = \\ &= \sum_{s=0}^{k-1} p^{k-q-s} x^{(s)}(0) ((1 - 1)^{k-s} - 1) = \\ &= - \sum_{s=0}^{k-1} p^{k-q-s} x^{(s)}(0), \end{aligned}$$

the formula (8.56) may be written in the form

$$\begin{aligned} q/q! x^{(k)}(t) &\equiv \sum_{s=0}^{q-k} \frac{1}{s!} \frac{d^s X(p)}{dp^s} \sum_{j=s}^q (-1)^j \frac{1}{p^{q-j}} \frac{1}{(j-s)!} (f^k)(j-s) + \\ &+ \sum_{s=q-k+1}^q \frac{1}{s!} \frac{d^s X(p)}{dp^s} \sum_{j=s}^q (-1)^j \frac{1}{p^{q-j}} \frac{1}{(j-s)!} (f^k)(j-s) = \\ &= \sum_{s=0}^{q-k} p^{s-q+k} \frac{1}{s!} \frac{d^s X(p)}{dp^s} \sum_{j=s}^{k+s} (-1)^j \binom{k}{j-s} + \\ &+ \sum_{s=q-k+1}^q p^{s-q+k} \frac{1}{s!} \frac{d^s X(p)}{dp^s} \sum_{j=s}^q (-1)^j \binom{k}{j-s} = \end{aligned}$$

$$\begin{aligned}
 &= \sum_{s=0}^{q-k} (-1)^s p^{s-q+k} \frac{1}{s!} \frac{d^s X(p)}{dp^s} \sum_{j=0}^k (-1)^j \binom{k}{j} + \\
 &+ \sum_{s=q-k+1}^q (-1)^s p^{s-q+k} \frac{1}{s!} \frac{d^s X(p)}{dp^s} \sum_{j=0}^{q-s} (-1)^j \binom{k}{j}.
 \end{aligned}$$

According to (8.2) and (8.3), we obtain

$$\begin{aligned}
 \frac{t^q}{q!} x^{(k)}(t) &\equiv \sum_{s=q-k+1}^q (-1)^s p^{s-q+k} \frac{1}{s!} \frac{d^s X(p)}{dp^s} (-1)^{s-q} \binom{k-1}{q-s} = \\
 &= (-1)^q \sum_{s=q-k+1}^q \binom{k-1}{q-s} p^{s-q+k} \frac{1}{s!} \frac{d^s X(p)}{dp^s}.
 \end{aligned}$$

Replacing here  $s$  with  $j$ , we obtain the formula (8.39) for  $q \geq k$ .

Substituting in (8.39) the suitable values of  $q$  and  $k$ , we obtain the formulas (8.41), ..., (8.54).

This completes the proof. •

(8.57) **Theorem.** If  $X(p, t)$ ,  $dX(p, t)/dt$ , ...,  $d^k X(p, t)/dt^k \in A$ , where  $k$  is a non-negative integer, then for any non-negative integer  $q$  we have

$$(8.58) \quad e^{qt} d^k X(p, t)/dt^k \equiv (p-q)^k e^{qt} X(p, t) - p \sum_{j=0}^{k-1} (p-q)^{k-j-1} [d^j X(p, t)/dt^j]_{t=0}.$$

In particular, we have

$$(8.59) \quad e^{qt} dX(p, t)/dt \equiv (p-q) e^{qt} X(p, t) - p X(p, 0),$$

$$(8.60) \quad e^{qt} d^2 X(p, t)/dt^2 \equiv (p-q)^2 e^{qt} X(p, t) - p(p-q) X(p, 0) - p[dX(p, t)/dt]_{t=0},$$

$$\begin{aligned}
 (8.61) \quad e^{qt} d^3 X(p, t)/dt^3 &\equiv (p-q)^3 e^{qt} X(p, t) - p(p-q)^2 X(p, 0) - \\
 &- p(p-q)[dX(p, t)/dt]_{t=0} - p[d^2 X(p, t)/dt^2]_{t=0}.
 \end{aligned}$$

**Proof.** We prove the theorem analogously to Theorem (8.20). •

(8.62) Theorem. If  $x(t)$ ,  $x'(t) := dx/dt$ , ...,  $x^{(n)}(t) := d^n x(t)/dt^n \in \mathbb{R}$ , where  $n$  is a non-negative integer, then the linear differential equation

$$(8.63) \quad \sum_{k=0}^n (a_k t + b_k) x^{(k)}(t) = f(t)$$

where  $f(t) \in \mathbb{R}$ , is equivalent to the linear differential equation

$$(8.64) \quad X'(p) \sum_{k=0}^n a_k p^k + X(p) \left( -a_0/p + \sum_{k=0}^{n-1} (k a_{k+1} - b_k) p^k - b_n p^n \right) \equiv \\ \equiv \sum_{k=1}^n p^k \left( \sum_{j=0}^{n-k-1} (k a_{k+j+1} - b_{k+j}) x^{(j)}(0) - b_n x^{(n-k)}(0) \right) - F(p),$$

where

$$x(t) \equiv X(p), \quad X'(p) := dX(p)/dp, \quad f(t) \equiv F(p).$$

Proof. By virtue of (8.41) and (8.55), the equation (8.63) is equivalent to

$$\sum_{k=0}^n \left( - (k-1) a_k p^{k-1} X(p) - a_k p^k X'(p) + a_k \sum_{j=0}^{k-2} (k-j-1) p^{k-j-1} x^{(j)}(0) \right) + \\ + \sum_{k=0}^n b_k p^k X(p) - \sum_{k=0}^n b_k \sum_{j=0}^{k-1} p^{k-j} x^{(j)}(0) \equiv F(p),$$

that is,

$$X'(p) \sum_{k=0}^n a_k p^k + X(p) \sum_{k=0}^n \left( (k-1) a_k p^{k-1} - b_k p^k \right) \equiv \\ \equiv \sum_{k=0}^n \left( a_k \sum_{j=0}^{k-2} (k-j-1) p^{k-j-1} x^{(j)}(0) - b_k \sum_{j=0}^{k-1} p^{k-j} x^{(j)}(0) \right) - F(p),$$

that is,

$$x(p) \sum_{k=0}^n a_k p^k + x(p) \left( -a_0/p + \sum_{k=0}^{n-1} (k a_{k+1} - b_{k+1}) p^k - b_n p^n \right) \equiv$$

$$\equiv \sum_{k=0}^n \left( a_k \sum_{j=1}^{k-1} p^j x^{(k-j-1)}(v) - b_k \sum_{j=1}^k p^j x^{(k-j)}(v) \right) - F(p),$$

which is equivalent to

$$x(p) \sum_{k=0}^n a_k p^k + x(p) \left( -a_0/p + \sum_{k=0}^{n-1} (k a_{k+1} - b_{k+1}) p^k - b_n p^n \right) \equiv$$

$$\equiv \sum_{k=2}^n \sum_{j=1}^{k-1} (j a_k x^{(k-j-1)}(v) - b_k x^{(k-j)}(v)) p^j - x(v) \sum_{k=1}^n b_k p^k - F(p),$$

that is,

$$x(p) \sum_{k=0}^n a_k p^k + x(p) \left( -a_0/p + \sum_{k=0}^{n-1} (k a_{k+1} - b_{k+1}) p^k - b_n p^n \right) \equiv$$

$$\equiv \sum_{j=1}^{n-1} p^j \sum_{k=j+1}^n (j a_k x^{(k-j-1)}(v) - b_k x^{(k-j)}(v)) - x(v) \sum_{k=1}^n b_k p^k - F(p),$$

that is,

$$x(p) \sum_{k=0}^n a_k p^k + x(p) \left( -a_0/p + \sum_{k=0}^{n-1} (k a_{k+1} - b_{k+1}) p^k - b_n p^n \right) \equiv$$

$$\equiv \sum_{k=1}^{n-1} p^k \sum_{j=k+1}^n (k a_j x^{(j-k-1)}(v) - b_j x^{(j-k)}(v)) - x(v) \sum_{k=1}^n b_k p^k - F(p),$$

that is,

$$\begin{aligned}
 & x'(p) \sum_{k=0}^n a_k p^k + x(p) \left( -a_0/p + \sum_{k=0}^{n-1} (k a_{k+1} - b_k) p^k - b_n p^n \right) \equiv \\
 & \equiv \sum_{k=t}^{n-1} p^k (k a_{k+1} x(0)) + \sum_{j=t}^{n-k-1} (k a_{k+j+1} - b_{k+j}) x^{(j)}(0) - b_n x^{(n-k)}(0) - \\
 & - x(0) \sum_{k=1}^n b_k p^k - F(p) ,
 \end{aligned}$$

which is equivalent to (8.64). •

(8.65) Example. Let us solve the differential equation

$$(8.66) \quad 2t x''' - (10t+1) x'' + (8t+15) x' - 20x = 0 ,$$

where  $x := x(t) \in \mathbb{R}$ .

According to Theorem (8.62), we obtain

$$(2p^3 - 10p^2 + 8p) X' + (5p^2 - 25p + 20) X \equiv 5p^2 x(0) + p(-25x(0) + 3x'(0)) ,$$

that is,

$$(8.67) \quad (2p^3 - 10p^2 + 8p) X' + (5p^2 - 25p + 20) X = 5p^2 x(0) + p(-25x(0) + 3x'(0)) + g(p) ,$$

where  $X = X(p) \equiv x(t)$ ,  $X' = dx/dp$  and  $g(p) \in \mathbb{R}$  is a function satisfying the condition

$$(8.68) \quad g(p) \equiv 0 .$$

Solving (8.67) by classical methods, we obtain

$$\begin{aligned}
 x &= x(0) + 1/p x'(0) - 5/p^2 (4x(0) - 3x'(0)) + H/(p^2\sqrt{p}) - \\
 &\quad - 1/5 (20x(0) - 3x'(0))/(p^2\sqrt{p}) \log \frac{\sqrt{p}+1}{\sqrt{p}-1} + \\
 &\quad + 1/3 (80x(0) - 48x'(0))/(p^2\sqrt{p}) \log \frac{\sqrt{p}+2}{\sqrt{p}-2} - \\
 &\quad - 1/(p^2\sqrt{p}) \int_p^\infty g(u) u\sqrt{u}/(2(u-1)(u-4)) du ,
 \end{aligned}$$

where  $H$  is an arbitrary constant.

Multiplying (8.68) on both sides by  $p\sqrt{p}$  and dividing by  $2(p-1)(p-4)$ , we see that

$$g(p)p\sqrt{p}/(2(p-1)(p-4)) \equiv 0$$

and, by Theorem (7.62),

$$\int_p^\infty g(u) u\sqrt{u}/(2(u-1)(u-4)) du \equiv 0 .$$

Dividing it on both sides by  $p^2\sqrt{p}$ , we obtain

$$1/(p^2\sqrt{p}) \int_p^\infty g(u)/(2\sqrt{p}(p-1)(p-4)) du \equiv 0 .$$

Hence we have

$$\begin{aligned}
 (8.69) \quad x &\equiv x(0) + 1/p x'(0) - 5/p^2 (4x(0) - 3x'(0)) + H/(p^2\sqrt{p}) - \\
 &\quad - 1/5 (20x(0) - 3x'(0))/(p^2\sqrt{p}) \log \frac{\sqrt{p}+1}{\sqrt{p}-1} + \\
 &\quad + 1/3 (80x(0) - 48x'(0))/(p^2\sqrt{p}) \log \frac{\sqrt{p}+2}{\sqrt{p}-2} .
 \end{aligned}$$

Using tables of Laplace Transforms and taking into account the difference between (1.1) and (1.2), we obtain

$$(8.70) \quad 1/p \equiv t, \quad 2/p^2 \equiv t^2, \quad 1/(p^2\sqrt{p}) \equiv 8/(15\sqrt{\pi}) t^2\sqrt{t} .$$

It remains to find the function  $y_\alpha(t) \equiv y_\alpha(p)$  for

$$(8.71) \quad y_\alpha(p) := 1/(p^2\sqrt{p}) \log \frac{\sqrt{p}+\alpha}{\sqrt{p}-\alpha}$$

and then to use it for  $\alpha = 1$  and  $\alpha = 2$ . We have

$$y_\alpha'(p) = -5/(2p^3\sqrt{p}) \log \frac{\sqrt{p}+\alpha}{\sqrt{p}-\alpha} - \frac{\alpha}{p^3(p-\alpha^2)}.$$

Hence

$$-p y_\alpha'(p) = 5/2 y_\alpha(p) - 1/\alpha^5 - 1/(\alpha^3 p) - 1/(\alpha p^2) + p \cdot (\alpha^5(p-\alpha^2)).$$

By virtue of (7.20), (5.4), (5.5) and Theorem (6.40), we have

$$\begin{aligned} t y_\alpha'(t) &= 5/2 y_\alpha(t) + \\ &+ 1/\alpha^5 (\alpha^2 t - 1 - \alpha^2 t - 1/2 \alpha^2 t^2). \end{aligned}$$

Solving this equation by classical methods, we obtain

$$\begin{aligned} y_\alpha(t) &= K_\alpha t^{2\sqrt{t}} + 2/(3\alpha^5) + 2/(3\alpha^3) t + 1/\alpha t^2 - \\ &- 2/(15\alpha^5) (3 + 2\alpha^2 t + 4\alpha^4 t^2) t^{2\sqrt{t}} + 16/15 t^{2\sqrt{t}} \int_0^{\alpha\sqrt{t}} d\tau \alpha \end{aligned}$$

as the function having the transform (8.71). We don't need to be concerned about the value of the constant  $K_\alpha$ , because the term  $K_\alpha t^{2\sqrt{t}}$  will be absorbed by the term  $A t^{2\sqrt{t}}$  in the final solution for  $x(t)$ .

Using this formula for  $\alpha = 1$  and  $\alpha = 2$  together with (8.70), we obtain from (8.69), in view of Theorem (6.40),

$$\begin{aligned} x(t) &= x(0) + x'(0) t - 5/2 (4x(0) - 3x'(0)) t^2 + 80/(15\sqrt{\pi}) t^{2\sqrt{t}} - \\ &- 1/6 (20x(0) - 3x'(0)) y_1(t) + \\ &+ 1/3 (80x(0) - 48x'(0)) y_2(t). \end{aligned}$$

Introducing

$$(8.72) \quad \begin{aligned} B &:= -4/9 x(0) + 1/15 x'(0), \\ C &:= 1/9 x(0) - 1/15 x'(0), \end{aligned}$$

and

$$A := 81/(15\sqrt{\pi}) + 15/2 BK_1 + 240 CK_2,$$

we obtain

$$(8.73) \quad x(t) = A t^2 \sqrt{t} - B(4t^2 + 2t + 3)e^t - C(54t^2 + 8t + 3)e^{4t} +$$

$$+ 8B t^2 \sqrt{t} \int_0^t e^{2u} du + 256C t^2 \sqrt{t} \int_0^t e^{4u} du,$$

where  $A, B, C$  are arbitrary constants. For any initial values  $x(0)$  and  $x'(0)$  constants  $B$  and  $C$  may be calculated from (8.72).

For example, for

$$x(0) = x'(0) = 0, \quad x(t) = t,$$

we have  $B = C = 0$ ,  $A = 1$  and, according to (8.73),

$$x(t) = t^2 \sqrt{t}.$$

(8.74) Example. Let us find a transform  $x = x(p)$  of the function

$$(8.75) \quad x = x(t) := e^{\frac{t^2}{2}} \in \mathcal{B}.$$

We have

$$x'(t) = t e^{\frac{t^2}{2}} = tx$$

and, by virtue of (6.63) and (7.19),

$$px - p \equiv 1/p x - x' ,$$

where  $x' := dx/dp$ , that is,

$$x' + (\rho - 1/p)x = \rho + g(\rho) ,$$

where  $g(\rho)$  is a function satisfying the condition

$$g(\rho) \equiv 0 .$$

Let us choose this transform  $x$  which corresponds to  $g(\rho) = 0$ , that is, satisfies the equation

$$(8.76) \quad x' + (\rho - 1/p)x = \rho .$$

Solving this equation by classical methods, we obtain

$$x = K\rho e^{-\frac{\rho^2}{2}} + \rho e^{-\frac{\rho^2}{2}} \int_0^\rho e^{\frac{u^2}{2}} du ,$$

where  $K$  is a constant. Since, according to Definition (6.1),

$$e^{\rho t} \rho e^{-\frac{\rho^2}{2}} \in A ,$$

it follows from Theorem (6.76) that

$$(8.77) \quad K\rho e^{-\frac{\rho^2}{2}} \equiv 0 .$$

Thus let us put  $K=0$ . Then

$$(8.78) \quad x = \rho e^{-\frac{\rho^2}{2}} \int_0^\rho e^{\frac{u^2}{2}} du .$$

Since we have (8.77) and the function  $e^{\frac{u^2}{2}}$  does not belong to the ring  $A$ , let us verify that such an  $x$  satisfies the condition  $x \in A$ . Since it satisfies evidently the conditions 1<sup>o</sup> and 2<sup>o</sup> from Definition (6.1), let us verify that it satisfies the condition 3<sup>o</sup> too. Let

$$y = e^{-\frac{\rho^2}{2}} \int_0^\rho e^{\frac{u^2}{2}} du .$$

Then we have

$$x = \rho y ,$$

$$y' = -\rho y + 1$$

and

$$|\gamma| = \int_0^p e^{-\frac{(x^2-y^2)}{2}} dx \leq \int_0^p 1 dx = p.$$

We shall prove by induction that

$$(8.79) \quad d^n x/dp^n = P_n Y + Q_n,$$

where  $P_n$  is a polynomial of degree  $n+1$  and  $Q_n$  a polynomial of degree  $n$ , defined as follows

$$P_0 = p, \quad Q_0 = 0$$

and for  $n = 1, 2, 3, \dots$

$$P_n = -p P_{n-1} + dP_{n-1}/dp, \quad Q_n = P_{n-1} + dQ_{n-1}/dp.$$

We see that (8.79) is true for  $n=0$ . Supposing that it is true for  $n=k$ , we have

$$\begin{aligned} d^{k+1} x/dp^{k+1} &= dP_k/dp Y + P_k dY/dp + dQ_k/dp = \\ &= dP_k/dp Y - p P_k Y + P_k + dQ_k/dp = \\ &= P_{k+1} Y + Q_{k+1}, \end{aligned}$$

which means that it is true for  $n=k+1$  too. Thus (8.79) is true for  $n=0, 1, 2, \dots$ .

Thus we have

$$|d^n x/dp^n| \leq |P_n| |\gamma| + |Q_n| \leq p |P_n| + |Q_n|.$$

Since for every non-negative integer  $n$  we have

$$p^n \leq n! e^p,$$

there exists a positive integer  $c_n$  such that

$$|d^n x/dp^n| \leq c_n e^p.$$

This means that  $x$  satisfies the condition 3 $\alpha$  from Definition (6.1). Thus we have  $x \in A$ . •

## IX. The distribution rings $\mathcal{D}$ and $\mathcal{D}_R$

If  $R = \{F(p, t)\} \in \mathcal{D}$  we write

$$-R = \{-F(p, t)\} .$$

If  $R_1 = \{F_1(p, t)\}, R_2 = \{F_2(p, t)\} \in \mathcal{D}$  we write

$$R_1 + R_2 = \{F_1(p, t) + F_2(p, t)\} ,$$

$$R_1 - R_2 = \{F_1(p, t) - F_2(p, t)\} ,$$

$$R_1 R_2 = \{F_1(p, t) \times F_2(p, t)\} .$$

If  $F(p, t) = 0$  we write

$$\{F(p, t)\} = 0 .$$

Similarly, if  $\theta = \{F(p, t)\} \in \mathcal{D}_R$  we write

$$-\theta = \{-F(p, t)\}$$

and if  $\theta_R = \{F_1(p, t)\}, \theta_R = \{F_2(p, t)\} \in \mathcal{D}_R$  we write

$$\theta_R + \theta_R = \{F_1(p, t) + F_2(p, t)\} ,$$

$$\theta_R - \theta_R = \{F_1(p, t) - F_2(p, t)\} ,$$

$$\theta_R \theta_R = \{F_1(p, t) \times F_2(p, t)\} .$$

If  $F(p, t) \equiv 0$  we write

$$\{F(p, t)\} = \emptyset .$$

The Theorem (7.10) allows us to introduce the following definition.

(9.1) Definition. The  $k$ -th derivative  $\mathcal{D}^k$  of an  $\mathcal{A}$ -distribution  $\mathcal{B} = \{F(p, t)\} \in D_{\mathcal{A}}$  is the  $\mathcal{A}$ -distribution

$$(9.2) \quad \mathcal{D}^k \mathcal{B} = \{D^k F(p, t)\} . \bullet$$

Here we put

$$\mathcal{D}^0 \mathcal{B} = \mathcal{B} .$$

Instead of  $\mathcal{D}^k \mathcal{B}$  we write also  $\mathcal{B}'$ , instead of  $\mathcal{D}^{k+1} \mathcal{B}$  also  $\mathcal{B}''$  etc.

(9.3) Theorem. We have for any  $\mathcal{A}$ -distributions  $\mathcal{B}, \mathcal{C} \in D_{\mathcal{A}}$

$$(9.4) \quad (\mathcal{B} + \mathcal{C})' = \mathcal{D}^1 \mathcal{B} + \mathcal{D}^1 \mathcal{C} ,$$

$$(9.5) \quad (\mathcal{B} - \mathcal{C})' = \mathcal{D}^1 \mathcal{B} - \mathcal{D}^1 \mathcal{C} ,$$

$$(9.6) \quad (\mathcal{B} \mathcal{C})' = \mathcal{B}' \mathcal{C} + \mathcal{B} \mathcal{C}' ,$$

$$(9.7) \quad (\mathcal{B} \mathcal{C})'' = \sum_{j=0}^k \binom{k}{j} \mathcal{D}^{k-j} \mathcal{B} \mathcal{D}^j \mathcal{C} .$$

$$(9.8) \quad (\mathcal{B}^k)' = k! \mathcal{B}^{k-1} .$$

Proof. The above formulas follow from Definition (9.1) and Theorems (7.4), (7.6) and (7.43). •

(9.9) Theorem. If  $\mathcal{B} = \{g(p)\} \in \mathcal{D}$  then

$$(9.10) \quad \mathcal{B}' = \{\partial g(p)/\partial p^j\} .$$

Proof. The theorem follows from Definition (9.1) and Theorem (7.46). •

(9.11) Theorem. The ring  $\mathcal{D}$  of distributions includes zero divisors.

Proof. It is sufficient to give an example of zero divisors. Let

$$\mathcal{B}_1 = \{g_1(p)\}, \quad \mathcal{B}_2 = \{g_2(p)\} \in \mathcal{D},$$

where

$$g_1(p) = \begin{cases} 0 & \text{for } 2n \leq \operatorname{Re} p < 2n+1, \\ 1 & \text{for } 2n+1 \leq \operatorname{Re} p < 2n+2; \end{cases}$$

$$g_2(p) = \begin{cases} 1 & \text{for } 2n \leq \operatorname{Re} p < 2n+1, \\ 0 & \text{for } 2n+1 \leq \operatorname{Re} p < 2n+2, \end{cases}$$

and  $n = 0, 1, 2, \dots$ . By virtue of Theorem (4.24) we have

$$\mathcal{B}_1 \mathcal{B}_2 = \{g_1(p) g_2(p)\} = \{0\} = 0 .$$

We shall show that none of the congruences

$$g_1(p) \equiv 0, \quad g_2(p) \equiv 0$$

is true. In order to do it, let us suppose that

$$(9.12) \quad g_j(p) \equiv 0 \quad (j = 1 \text{ or } j = 2) .$$

In view of Theorem (4.47), we have

$$e^{at} g_j(p) \in F .$$

This means, by virtue of Theorem (3.3), that there exist real numbers  $a, b$ ,  $0 \leq a < t$ ,  $b \geq 0$ , and a function  $c(t) \in C$ ,  $c(t) \neq 0$ , such that

$$e^{-(at+b)p} \int_0^t c(t-u) e^{bu} g_j(p) du \Rightarrow 0 , \quad p \rightarrow \infty$$

that is,

$$g_j(p) \int_0^{at+b} c(t-u) e^{p(u-at-b)} du + g_j(p) \int_{at+b}^t c(t-u) e^{p(u-at-b)} du \Rightarrow 0 . \quad p \rightarrow \infty$$

Since the function  $c(t)$  is bounded in every finite interval  $0 \leq t \leq T$ , we have

$$g_j(p) \int_0^{at+b} c(t-u) e^{p(u-at-b)} du \Rightarrow 0 . \quad p \rightarrow \infty$$

Therefore for every  $t \geq b/(1-a)$

$$g_j(p) \int_{at+b}^{t-at-b} c(t-u) e^{p(u-at-b)} du = g_j(p) \int_0^{t-at-b-v} c(t-at-b-v) e^{pv} dv \Rightarrow 0 , \quad p \rightarrow \infty$$

that is, for every  $t \geq 0$

$$(9.13) \quad g_j(p) \int_0^t c(t-u) e^{pu} du \Rightarrow 0 . \quad p \rightarrow \infty$$

Let

$$r_n := \begin{cases} 2n+1 & \text{for } j=1 , \\ 2n & \text{for } j=2 . \end{cases}$$

Since  $g_j(r_n) = 1$ , (9.13) may be written in the form

$$\int_0^t e^{2nu} e^u c(t-u) du \Rightarrow 0 \quad \text{for } j=1 , \quad n \rightarrow \infty$$

$$\int_0^t e^{2nu} c(t-u) du \Rightarrow 0 \quad \text{for } j=2 , \quad n \rightarrow \infty$$

which is equivalent to

$$\int_0^{2t} e^{ju/2} c(t-u/2) du \underset{n \rightarrow \infty}{\Rightarrow} 0 \quad \text{for } j = 1,$$

$$\int_0^{2t} e^{ju/2} c(t-u/2) du \underset{n \rightarrow \infty}{\Rightarrow} 0 \quad \text{for } j = 2.$$

By virtue of the Theorem on Bounded Moments (see [5], p.18), we have

$$e^{ju/2} c(t-u/2) = 0 \quad \text{for } j = 1,$$

$$c(t-u/2) = 0 \quad \text{for } j = 2,$$

for  $0 \leq u \leq 2t$ , that is, in both cases,  $c(u) = 0$  for  $0 \leq u \leq t$ . Since  $t$  can be arbitrarily great, we obtain  $c(t) = 0$ , which contradicts the assumption that  $c(t) \neq 0$ . This means that (9.12) is not true. It completes the proof. •

It remains an open problem whether the distribution ring  $\mathcal{D}_g$  has zero divisors.

Since the ring  $\mathcal{D}$  has zero divisors, we cannot construct in the ordinary way a quotient field of distributions from the ring  $\mathcal{D}$ . But in the following chapter we shall construct such a field for a subring of  $\mathcal{D}$ .

## X. The quotient field $\mathbb{Q}$

(10.1) Definition. A function  $F(p, t) \in S$  will be called a **quotient function** iff there exist functions  $h(t), k(t) \in \mathcal{L}$ ,  $k(t) \neq 0$ , such that

$$(10.2) \quad F(p, t) \times k(t) \equiv h(t).$$

In such a case we shall write

$$(10.3) \quad F(p, t) \equiv \frac{h(t)}{k(t)}.$$

(10.4) Theorem. If  $F(p, t)$  is a quotient function then  $F(p, t) \in F$ .

Proof. The theorem follows from Definition (10.1) and Theorem (3.20). •

(10.5) Theorem. If  $n$  is an arbitrary integer and  $f(t) \in S$ , then  $p^n f(t)$  is a quotient function. In particular, every  $f(t) \in S$  is a quotient function.

Proof. If  $n$  is a non-negative integer, the theorem follows from the fact that, according to (5.4), we have

$$p^n f(t) \times t^{n+1}/(n+1)! = p^n f(t) \times 1/p^{n+1} = f(t) \times 1/p = \int_0^t f(u) du$$

which means that  $p^n f(t)$  is a quotient function. Hence for  $n=0$  we obtain the particular case.

If  $n$  is negative then, putting  $n := -n$ , we have

$$\begin{aligned} p^n f(t) &= 1/p^n f(t) = 1/p f(t) \times 1/p^{n-1} = 1/p f(t) \times t^{n-1}/(n-1)! = \\ &= \int_0^t f(t-u) u^{n-1}/(n-1)! du \end{aligned}$$

which means that  $p^n f(t)$  is a quotient function. This completes the proof. •

(10.6) Theorem. If

$$F(p, t) \equiv \frac{h(t)}{k(t)}$$

and

$$F(p, t) \equiv g(p, t) \in F$$

then  $g(p, t)$  is also a quotient function and

$$g(p, t) \equiv \frac{h(t)}{k(t)} .$$

Proof. If  $F(p, t) = g(p, t)$  then, according to (4.19), the congruence (10.2) implies

$$g(p, t) \times k(t) \equiv h(t) .$$

Hence we obtain our assertion. •

(10.7) Theorem. If we have (10.3) and  $H(p, t), K(p, t) \in F$  are functions such that

$$(10.8) \quad H(p, t) = h(t), \quad K(p, t) = k(t),$$

then  $H(p, t)$  and  $K(p, t)$  are also quotient functions and

$$(10.9) \quad F(p, t) \times K(p, t) \equiv H(p, t),$$

which will be written in the form

$$(10.10) \quad F(p, t) = \frac{H(p, t)}{K(p, t)},$$

analogous to (10.3).

Proof.  $H(p, t)$  and  $K(p, t)$  are quotient functions by virtue of (10.8) and Theorems (10.5) and (10.6). Accordingly to (4.19), it follows from (10.8) and (10.2) that

$$F(p, t) \times K(p, t) \equiv F(p, t) \times k(t) \equiv h(t) \equiv H(p, t),$$

that is (10.9). •

(10.11) Theorem. The set of all quotient functions forms a subring of the ring  $F$ .

Proof. Let  $F_1(p, t)$  and  $F_2(p, t)$  be quotient functions. Then there exist functions  $h_j(t), k_j(t) \in \mathcal{L}$ ,  $k_j(t) \neq 0$ , ( $j = 1, 2$ ) such that

$$(10.12) \quad F_1(p, t) \times k_j(t) \equiv h_j(t), \quad F_2(p, t) \times k_j(t) \equiv h_2(t).$$

Multiplying the first congruence by  $1/p k_2(t)$  and the second one by  $1/p k_1(t)$  and adding the congruences thus obtained, we have

$$\begin{aligned} (F_1(p, t) + F_2(p, t)) \times (1/p k_1(t) \times k_2(t)) &= \\ &= 1/p h_1(t) \times k_2(t) + 1/p h_2(t) \times k_1(t). \end{aligned}$$

By virtue of Theorem (2.27)

$$(10.13) \quad k(t) := \int_0^t k_1(t-u) k_2(u) du \in \mathcal{L}, \quad k(t) \neq 0.$$

Thus,  $F_1(p, t) + F_2(p, t)$  is a quotient function.

If  $F(p, t)$  is a quotient function, so is  $-F(p, t)$ .

Now, multiplying the congruences (10.12) we obtain

$$F_1(p, t) \times F_2(p, t) \times k_1(t) \times k_2(t) \equiv h_1(t) \times h_2(t)$$

and also

$$(F_1(p, t) \times F_2(p, t)) \times (1/p k_1(t) \times k_2(t)) \equiv (1/p h_1(t) \times h_2(t))$$

which means, by (10.13), that  $F_1(p, t) \times F_2(p, t)$  is a quotient function. Thus the proof is complete. •

(10.14) Definition. The subring of all quotient functions will be called the ring  $\mathcal{F}_q$ . •

(10.15) Theorem. If  $F(p, t) \in \mathcal{F}_q$ ,  $\theta(p, t) \in \mathcal{F}$ , and

$$F(p, t) \times \theta(p, t) = 0,$$

then

$$F(p, t) \equiv 0 \quad \text{or} \quad \theta(p, t) \equiv 0.$$

Proof. It is sufficient to prove that, if the congruence  $F(p, t) \equiv 0$  is not true, then  $\theta(p, t) \equiv 0$ . By assumption, there exist functions  $h(t)$ ,  $k(t) \in \mathcal{L}$ ,  $k(t) \neq 0$ , such that

$$F(p, t) \times k(t) = h(t).$$

If  $h(t) \equiv 0$ , then, by Theorem (4.40),  $F(p, t) \equiv 0$ . It follows that, if the congruence  $F(p, t) \equiv 0$  is not true, then  $h(t) \neq 0$ , and, by Theorem (4.42),  $\theta(p, t) \equiv 0$ , which was to be proved. •

(10.16) Theorem. The ring  $F_q$  has no zero divisors.

Proof. The theorem follows immediately from Theorem (10.15). •

(10.17) Definition. If  $F(p, t) \in F_q$  and  $R = \{F(p, t)\}$  then  $R$  will be called a  $q$ -distribution, which in the case of (10.10) will be written in the form

$$(10.18) \quad R = \{F(p, t)\} = \frac{\{H(p, t)\}}{\{K(p, t)\}} . \bullet$$

(10.19) Theorem. If we have (10.18) and  $F^*(p, t), H^*(p, t), K^*(p, t) \in F$  are functions such that

$$F^*(p, t) \equiv F(p, t) , \quad H^*(p, t) \equiv H(p, t) , \quad K^*(p, t) \equiv K(p, t) ,$$

then

$$R = \{F^*(p, t)\} = \frac{\{H^*(p, t)\}}{\{K^*(p, t)\}} .$$

Proof. The theorem follows from Theorems (10.6) and (10.7). •

(10.20) Theorem. All representatives of a  $q$ -distribution are quotient functions.

Proof. The theorem follows immediately from Theorem (10.6). •

(10.21) Theorem. All q-distributions form a subring of the ring  $\mathbb{B}$ .

Proof. The theorem follows immediately from Definition (10.17) and Theorem (10.11). •

(10.22) Definition. The subring of all q-distributions will be called the ring  $D_q$ . •

(10.23) Definition. A sequence of q-distributions  $R_1, R_2, \dots$  will be called convergent to a q-distribution  $R$  and it will be written

$$\lim_{n \rightarrow \infty} R_n = R$$

iff there are functions  $h(t), f_1(t), f_2(t), \dots \in \mathcal{L}$ ,  $h(t) \neq 0$ , such that

$$10 \quad R_a = \begin{cases} r_a(t) \\ h_a(t) \end{cases} \quad \text{for } a = 1, 2, \dots ,$$

2º the sequence  $r_1(t), r_2(t), \dots$  converges uniformly in every finite interval  $0 \leq t \leq T$  to a function  $r(t) \in \mathcal{E}$  (we write  $r_n(t) \rightarrow r(t)$ ).

$$R = \begin{pmatrix} f(t) \\ h(t) \end{pmatrix} . \quad (*)$$

(10.24) Theorem. If  $A_1, A_2, \dots$  is any sequence of q-distributions and  $\lim A_n = A$  and  $\lim A'_n = A'$  then

$$R = \text{[REDACTED]}.$$

(\*) The idea of a convergent sequence of  $q$ -distributions is analogous to that used by J.Mikusinski for convergent sequences of operators ([3], p.41). See also [4], p.311-333.

Proof. By assumption, there exist in the class  $\mathcal{C}$  functions  $h(t) \neq 0$ ,  $k(t) \neq 0$ ,  $r_n(t)$ ,  $g_n(t)$  ( $n = 1, 2, \dots$ ),  $f(t)$ ,  $g(t)$ , such that

$$(10.25) \quad R_n\{h(t)\} = \{r_n(t)\} \quad \text{and} \quad R_n\{k(t)\} = \{g_n(t)\} ,$$

$$(10.26) \quad \underset{n \rightarrow \infty}{\lim} r_n(t) = f(t) \quad \text{and} \quad \underset{n \rightarrow \infty}{\lim} g_n(t) = g(t) , \quad \text{and}$$

$$(10.27) \quad R\{h(t)\} = \{f(t)\} \quad \text{and} \quad R^*\{k(t)\} = \{g(t)\} .$$

From (10.25) we obtain

$$R_n\{h(t)\}\{k(t)\} = \{k(t)\}\{r_n(t)\} = \{h(t)\}\{g_n(t)\}$$

and then

$$\{k(t) \times r_n(t)\} = \{h(t) \times g_n(t)\}$$

which means

$$k(t) \times r_n(t) = h(t) \times g_n(t) .$$

Hence

$$(10.28) \quad 1/p \ k(t) \times r_n(t) = 1/p \ h(t) \times g_n(t) ,$$

that is,

$$\int_0^t k(t-u) r_n(u) du = \int_0^t h(t-u) g_n(u) du .$$

By virtue of Theorem (4.39) the congruence (10.28) implies

$$1/p \ k(t) \times r_n(t) = 1/p \ h(t) \times g_n(t) .$$

By (10.26) we obtain

$$(10.29) \quad 1/p \ k(t) \times f(t) = 1/p \ h(t) \times g(t) .$$

If there are such functions  $F(p, t)$ ,  $G(p, t) \in F$  that

$$R = \{F(p, t)\} \quad \text{and} \quad R^* = \{G(p, t)\}$$

then, by (10.27)

$$F(p, t) \times h(t) = f(t), \quad G(p, t) \times k(t) = g(t)$$

and, further,

$$F(p, t) \times 1/p h(t) \times k(t) \equiv 1/p k(t) \times f(t),$$

$$G(p, t) \times 1/p h(t) \times k(t) \equiv 1/p h(t) \times g(t).$$

By virtue of (10.29) we obtain

$$(F(p, t) - G(p, t)) \times (1/p h(t) \times k(t)) \equiv 0$$

and by Theorem (4.40)

$$F(p, t) - G(p, t) \equiv 0.$$

It means, that  $R = R^*$ , which was to be proved. •

(10.30) Theorem. If  $R_1, R_2, \dots$  and  $S_1, S_2, \dots$  are any sequences of q-distributions and

$$\lim_{n \rightarrow \infty} R_n = R, \quad \lim_{n \rightarrow \infty} S_n = S,$$

then

$$(10.31) \quad \lim_{n \rightarrow \infty} (R_n + S_n) = R + S,$$

$$(10.32) \quad \lim_{n \rightarrow \infty} (R_n - S_n) = R - S,$$

$$(10.33) \quad \lim_{n \rightarrow \infty} R_n S_n = R S.$$

Proof. By assumption, there are in the class  $\mathcal{L}$  functions  $h(t) \neq 0$ ,  $k(t) \neq 0$  and  $f_a(t), g_a(t)$  ( $a = 1, 2, \dots$ ) such that

$$(10.34) \quad R_m\{h(t)\} = \{r_m(t)\}, \quad S_m\{k(t)\} = \{g_m(t)\},$$

$$(10.35) \quad \begin{matrix} r_m(t) \equiv r(t), \\ m \rightarrow \infty \end{matrix}, \quad \begin{matrix} g_m(t) \equiv g(t), \\ m \rightarrow \infty \end{matrix}$$

$$(10.36) \quad R\{h(t)\} = \{r(t)\}, \quad S\{k(t)\} = \{g(t)\}.$$

From (10.34) we obtain

$$R_m\{h(t)\}\{k(t)\} = \{k(t)\}\{r_m(t)\}, \quad S_m\{h(t)\}\{k(t)\} = \{h(t)\}\{g_m(t)\}$$

and

$$(R_m + S_m)\{1/p \ h(t) \times k(t)\} = \{1/p \ k(t) \times r_m(t) + 1/p \ h(t) \times g(t)\}.$$

On the other hand, from (10.36) we obtain analogously

$$(R + S)\{1/p \ h(t) \times k(t)\} = \{1/p \ k(t) \times r(t) + 1/p \ h(t) \times g(t)\}.$$

Since by (10.35)

$$\begin{aligned} 1/p \ k(t) \times r_m(t) + 1/p \ h(t) \times g_m(t) &= \\ &= \int_0^t k(t-u) r_m(u) du + \int_0^t h(t-u) g_m(u) du \underset{m \rightarrow \infty}{\equiv} \\ &\underset{m \rightarrow \infty}{\equiv} \int_0^t k(t-u) r(u) du + \int_0^t h(t-u) g(u) du = \\ &= 1/p \ k(t) \times r(t) + 1/p \ h(t) \times g(t) \end{aligned}$$

then, by definition, we obtain (10.31).

We prove (10.32) analogously.

In order to prove (10.33), we multiply the equalities (10.34) and obtain

$$R_m S_m\{h(t)\}\{k(t)\} = \{r_m(t)\}\{g_m(t)\}$$

and hence

$$R_B \{ \int_0^t h(t-u) k(u) du \} = \{ \int_0^t f_B(t-u) g_B(u) du \} .$$

Analogously, we have by (10.36)

$$R_B \{ \int_0^t h(t-u) k(u) du \} = \{ \int_0^t f(t-u) g(u) du \} .$$

Since

$$\int_0^t f_B(t-u) g_B(u) du \underset{n \rightarrow \infty}{\equiv} \int_0^t f(t-u) g(u) du$$

then we obtain (10.33). This completes the proof. •

(10.37) Definition. A sequence of quotient functions  $F_1(p, t), F_2(p, t), \dots$  will be called **distributionally convergent** to a quotient function  $F(p, t)$  and it will be written

$$\lim_{n \rightarrow \infty} \text{dis } F_n(p, t) = F(p, t)$$

iff

$$\lim_{n \rightarrow \infty} \{F_n(p, t)\} = \{F(p, t)\} . \quad \bullet$$

(10.38) Definition. A series of q-distributions  $R_1 + R_2 + \dots$  will be called **convergent to a q-distribution  $R$**  and it will be written

$$R = R_1 + R_2 + \dots$$

iff

$$R = \lim_{n \rightarrow \infty} (R_1 + R_2 + \dots + R_n) . \quad \bullet$$

(10.39) Definition. A series of quotient functions  $F_1(p, t) + F_2(p, t) + \dots$  will be called **distributionally convergent** to a quotient function  $F(p, t)$  and it will be written

$$F(p, t) \equiv F_1(p, t) + F_2(p, t) + \dots$$

iff

$$F(p, t) \equiv \lim_{n \rightarrow \infty} \text{dis} (F_1(p, t) + F_2(p, t) + \dots + F_n(p, t)) . \bullet$$

(10.40) Example. Although the sequence

$$\rho t^{1/1!}, \rho^2 t^2/2!, \rho^3 t^3/3!, \dots$$

is for every  $\rho$  uniformly convergent to zero with respect to  $t$  in every finite interval  $0 \leq t \leq r$ , we have

$$(10.41) \quad \lim_{n \rightarrow \infty} \{\rho^n t^n/n!\} = \{1\}, \quad \text{i.e.} \quad \lim_{n \rightarrow \infty} \text{dis} \rho^n t^n/n! \equiv 1 .$$

Indeed, since

$$\{\rho^{n+1} t^{n+1}/(n+1)!\} \{1\} = \{\rho^{n+1} t^{n+1}/(n+1)!\}$$

and by (5.4)

$$\rho^{n+1} t^{n+1}/(n+1)! \equiv 1$$

then

$$\{\rho^n t^n/n!\} \{1\} = \{1\}$$

and for  $n=0$

$$\{1\} \{1\} = \{1\} .$$

It means (10.41), by definition. •

(10.42) Theorem. If  $r_0(t), r_1(t), r_2(t), \dots \in \mathcal{C}$  and

$$(10.43) \quad \lim_{n \rightarrow \infty} r_n(t) = r(t)$$

then

$$(10.44) \quad \lim_{n \rightarrow \infty} \text{dis } r_n(t) = r(t)$$

that is,

$$(10.45) \quad \lim_{n \rightarrow \infty} \{r_n(t)\} = \{r(t)\} .$$

(See [2], p.41).

Proof. Since

$$\{r_n(t)\} \{t\} = \left\{ \rho \int_0^t r_n(u) du \right\}$$

and, by Theorem (4.20),

$$(10.46) \quad \{r_n(t)\} \{t\} = \{r_n(t)\} , \quad \{r(t)\} \{t\} = \{r(t)\}$$

then, by definition, (10.45) results from (10.43). Since (10.45) is equivalent to (10.44), the proof is complete. •

(10.47) Convention. Accordingly to (10.46) we can write simply

$$\{t\} = t .$$

Thus, for every distribution  $A$  we have

$$t \cdot A = A \cdot t = A .$$

Generally, for every real number  $r$  we shall write

$$\{r\} = r . \quad *$$

(10.48) Remark. There exist sequences of functions from the class  $\mathcal{L}$ , which are divergent in the usual sense, but which are distributionally convergent. •

(10.49) Example. (See [2], p.41). The sequence

$$\sin t, 2\sin 2t, 3\sin 3t, \dots$$

is divergent in the usual sense, but

$$(10.50) \quad \lim_{n \rightarrow \infty} \operatorname{dis} n \sin nt = p .$$

Indeed, we have

$$\begin{aligned} n \sin nt &= p \int_0^t n \sin nu du = p(t - \cos nt) = \\ &= p \int_0^t p(t - \cos nu) du = p^2(t - 1/n \sin nt) \end{aligned}$$

then, by (5.4),

$$1/p^2 = t^2/2$$

and, thus,

$$\{n \sin nt\}\{t^2/2\} = \{p^2(t - 1/n \sin nt)\}\{1/p^2\} = \{t - 1/n \sin nt\} .$$

Since

$$t - 1/n \sin nt \rightrightarrows t$$

and, by (5.4),

$$\{\rho\}\{t^2/2\} = \{\rho\}\{1/p^2\} = \{1/p\} = \{t\}$$

then

$$\lim_{n \rightarrow \infty} \{ n \sin n\ell \} = \{ \rho \}$$

that is, (10.50). •

(10.51) Theorem. The ring  $B_g$  has no zero divisors but it has an unit element 1.

Proof. The theorem follows from Definition (10.17), Theorem (10.16) and Convention (10.47). •

(10.52) Definition. A quotient distribution is a pair of q-distributions  $R, S \in B_g$ , where  $S \neq 0$  (in the sense of Convention (10.47)), written in the form

$$(10.53) \quad \frac{R}{S}$$

with arithmetical rules defined as follows

$$(10.54) \quad \frac{R}{S} = \frac{C}{D} \quad \text{iff} \quad RD = SC ,$$

$$(10.55) \quad \frac{R}{S} + \frac{C}{D} = \frac{RD + SC}{SD} ,$$

$$(10.56) \quad \frac{R}{S} \cdot \frac{C}{D} = \frac{RC}{SD} ,$$

$$(10.57) \quad \frac{R}{S} : \frac{C}{D} = \frac{RD}{SC} \quad \text{if } C \neq 0 ,$$

$$(10.58) \quad \frac{R}{1} = R .$$

(10.59) Theorem. The set of all quotient distributions forms a field.

Proof. The theorem follows immediately from Definition (10.52). •

(10.60) Definition. The set of all quotient distributions with arithmetical rules (10.54) - (10.58) will be called the field  $\mathcal{Q}$ . •

(10.61) Theorem. Every q-distribution belongs to  $\mathcal{Q}$ .

Proof. Every q-distribution belongs to  $\mathcal{Q}$  in the sense of isomorphism established by Conventions (10.47) and (10.58). •

(10.62) Theorem. Every quotient distribution can be written in the form

$$(10.63) \quad \frac{\{ f(t) \}}{\{ g(t) \}}$$

where  $f(t), g(t) \in \mathcal{E}, g(t) \neq 0$ .

Proof. Let

$$(10.64) \quad \frac{A}{B}$$

be any quotient distribution, where

$$A = \{ F(p, t) \} , \quad B = \{ G(p, t) \} \neq 0 ,$$

and  $F(p, t), G(p, t)$  are quotient functions. Hence there exist functions  $a(t), b(t), c(t), d(t) \in \mathcal{E}, b(t) \neq 0, c(t) \neq 0, d(t) \neq 0$ , such that

$$F(p, t) \times b(t) \equiv a(t) , \quad G(p, t) \times d(t) \equiv c(t) .$$

Multiplying the numerator and the denominator of (10.64) by

$$\{b(t) \times d(t)\} = \{b(t)\}\{d(t)\}$$

we obtain

$$\begin{aligned} \frac{a}{b} &= \frac{\{F(p, t)\}\{b(t)\}\{d(t)\}}{\{G(p, t)\}\{b(t)\}\{d(t)\}} = \frac{\{F(p, t) \times b(t)\}\{d(t)\}}{\{G(p, t) \times d(t)\}\{b(t)\}} = \frac{\{a(t)\}\{d(t)\}}{\{c(t)\}\{b(t)\}} = \\ &= \frac{\{1/p\}\{a(t)\}\{d(t)\}}{\{1/p\}\{c(t)\}\{b(t)\}} = \frac{\{r(t)\}}{\{g(t)\}} \end{aligned}$$

where

$$r(t) := \int_0^t a(t-u) d(u) du , \quad g(t) := \int_0^t c(t-u) b(u) du$$

which completes the proof. •

It is an open problem, whether such elements (10.64) exist which are not q-distributions. That such a case would be rare, we can see from the following theorem.

(10.65) Theorem. If

$$\frac{\{r(t)\}}{\{g(t)\}} \in Q$$

and the function  $g(t)$  has a transform  $h(p)$  such that

$$\frac{1}{h(p)} \in F$$

then there exists a quotient function  $F(p, t)$  such that

$$\{F(p, t)\} = \frac{\{fct\}}{\{g(t)\}}.$$

Proof. By definition of a transform, we have

$$\{g(t)\} = \{h(p)\}$$

and thus

$$\frac{\{fct\}}{\{g(t)\}} = \frac{\{fct\}}{\{h(p)\}}.$$

Putting

$$F(p, t) := \frac{t}{h(p)} f(t) \in F$$

we obtain

$$F(p, t) \times g(t) \equiv F(p, t) \times h(p) \equiv h(p) F(p, t) = f(t).$$

Thus,  $F(p, t)$  is a quotient function and

$$\{F(p, t)\}\{g(t)\} = \{fct\}.$$

Hence

$$\frac{\{fct\}}{\{g(t)\}} = \frac{\{F(p, t)\}\{g(t)\}}{\{g(t)\}} = \{F(p, t)\}$$

which was to be proved. •

It is also an open problem, whether such a function  $g(t) \in \mathcal{L}$ ,  $g(t) \neq 0$ , exists, which has transforms, but no transform  $h(p)$  such that

$$(10.66) \quad \frac{t}{h(p)} \in F.$$

In other words, it is an open problem, whether the conditions (10.66) and  $g(t) \neq 0$  are equivalent. We know, of course, that  $g(t) = 0$  implies that (10.66) is not true, but not conversely.

## XI. A theory corresponding to the Laplace Transformation

Let us consider, how to change the theory presented above, in order to obtain a correspondence to the Laplace Transformation but not to the Laplace-Carson Transformation. First, let us notice, that the function  $f(t) = 1$  corresponds by the Laplace Transformation to the transform  $g(s) = 1/s$  and not to  $1$ , as it was by the Laplace-Carson Transformation. Thus, we must construct the new theory in such a way, as to distinguish  $f(t) = 1$  and  $g(s) = 1$ . (According to custom, we replace the variable  $\rho$  used in the theory of the Laplace-Carson Transformation by the variable  $s$  used for the Laplace Transformation).

We shall deal with pairs of feasible functions, in the sense of Definition (3.1) or in the narrower sense of Definition (6.1), namely

$$(11.1) \quad [F(s, t), g(s)] .$$

The arithmetical rules, however, are now as follows:

$$(11.2) \quad [F_1(s, t), g_1(s)] + [F_2(s, t), g_2(s)] = \\ = [F_1(s, t) + F_2(s, t), g_1(s) + g_2(s)] ,$$

$$(11.3) \quad [F_1(s, t), g_1(s)] \times [F_2(s, t), g_2(s)] = \\ = \left[ \int_0^t F_1(s, t-u) F_2(s, u) du + g_1(s) F_2(s, t) + g_2(s) F_1(s, t), g_1(s) g_2(s) \right].$$

The above arithmetical operations are associative and commutative, multiplication is distributive with respect to addition.

By virtue of isomorphism, we introduce the following notations:

$$[F(s, t), \theta] = \langle F(s, t) \rangle, \quad [\theta, g(s)] = [g(s)].$$

Thus, by definition of addition,

$$[F(s, t), g(s)] = \langle F(s, t) \rangle + [g(s)].$$

By definition of multiplication, we have

$$(11.4) \quad \langle F_1(s, t) \rangle \times \langle F_2(s, t) \rangle = \left\langle \int_0^t F_1(s, t-u) F_2(s, u) du \right\rangle, \\ [g_1(s)] \times [g_2(s)] = [g_1(s) g_2(s)].$$

More generally,

$$[h(s)] \times [F(s, t), g(s)] = [h(s) F(s, t), h(s) g(s)].$$

Thus, we have

$$[F_1(s, t), g_1(s)] \times [F_2(s, t), g_2(s)] = \\ = (\langle F_1(s, t) \rangle + [g_1(s)]) \times (\langle F_2(s, t) \rangle + [g_2(s)]) = \\ = \langle F_1(s, t) \rangle \times \langle F_2(s, t) \rangle + [g_1(s)] \times \langle F_2(s, t) \rangle + \\ + \langle F_1(s, t) \rangle \times [g_2(s)] + [g_1(s)] \times [g_2(s)].$$

We have also

$$\langle f \rangle \times [F(s, t), g(s)] = [F(s, t), g(s)].$$

The set of all pairs (11.1) forms a commutative ring, which will be called the **ring Z**. The set of all functions  $\langle F(s, t) \rangle$  forms a commutative ring with multiplication (11.4) and will be called the **ring H**. Theorems (3.20), (3.24) and (3.26) hold also for the ring **H**.

Definition (4.1) changes as follows:

The **class Z** is the set of all pairs  $[F(s, t), \theta] \in L$  satisfying the condition

$$\langle e^{st} \times F(s, t) \rangle \in \mathbb{H} .$$

We introduce congruences in the same way as in Chapter IV, with an additional congruence

$$\langle s g(s) \rangle \equiv [g(s)] .$$

Then we have

$$\begin{aligned} \langle g(s) \rangle \times \langle F(s, t) \rangle &\equiv [1/s g(s)] \times \langle F(s, t) \rangle = \\ &= \langle 1/s g(s) F(s, t) \rangle . \end{aligned}$$

The propositions of Theorems (4.75) and (4.78) change as follows

$$\begin{aligned} \langle d/dt F(s, t) \rangle &\equiv \langle s F(s, t) \rangle - [F(s, 0)] , \\ \langle d^k/dt^k F(s, t) \rangle &\equiv \langle s^k F(s, t) \rangle - [s^{k-1} F(s, 0)] - \dots - [s^{k-1} F(s, 0)] . \end{aligned}$$

The propositions of Theorems (4.81) and (4.87) can be written in the form

$$\begin{aligned} \langle F(s, t) \rangle &\equiv \langle s \int_0^\infty e^{-st} F(s, t) dt \rangle \equiv \left[ \int_0^\infty e^{-st} F(s, t) dt \right] , \\ \langle f(t) \rangle &\equiv \langle s \int_0^\infty e^{-st} f(t) dt \rangle \equiv \left[ \int_0^\infty e^{-st} f(t) dt \right] . \end{aligned}$$

There are many problems connected with the theory presented here, which were not considered. The aim of this chapter is to give only an outline of the whole theory.

## XII. Final considerations

In this chapter we shall consider the relationship of the above-presented theory to other existing theories and discuss advantages resulting from it. We shall also point out some open problems.

First of all we shall consider the relationship to the theory and uses of the Laplace Transformation (or the Laplace-Carson Transformation), because this is the most popular basis for Operational Calculus. We have already made some general remarks in our Introduction, but let us now consider the problem once more in a deeper way.

The most important statement is, that the theory presented in this paper includes the whole theory of the Laplace Transformation (exactly speaking the theory of the Laplace-Carson Transformation and the theory of Laplace Transformation in the sense of Chapter XI). Thus, we cannot lose any advantage of the Laplace Transformation, but we can gain some new possibilities, not available before. The fundamental theorem, enabling us to use everything from the theory and applications of the Laplace Transformation, is Theorem (4.87). Moreover, it was proved, that the Laplace formula (4.88) may be generalized for functions of two variables  $F(p, t)$  in the sense of Theorem (4.81). The main disadvantage of the Laplace Transformation is, that in solving differential equations, we obtain as solutions only functions transformable in the sense of Laplace, and we are not able to get solutions from a larger class of functions. This observation was the starting-point for J.G.Mikusinski to construct his theory of Operational Calculus. The theory presented above starts from the same larger class of functions as Mikusinski did. But he lost the possibility of including the theory and uses of the Laplace Transformation. For example, he had much trouble in adapting some facts from that theory to his operators. The theory, presented above, includes all results of the Laplace Transform theory.

It is a well-known fact that Laplace transforms are analytic functions in half-planes  $p > p_0$ . This was the starting-point for introducing in our

theory a smaller but simpler ring  $\mathcal{A}$  of functions  $F(p, t)$  in Chapter VI. This ring is smaller with respect to the variable  $p$  only, not reducing the class of feasible functions of the variable  $t$ , which means that it does not reduce solutions of differential equations, solvable by Operational Calculus. For most applications this ring  $\mathcal{A}$  is quite sufficient. Some important theorems, like Theorem (6.29), hold for the ring  $\mathcal{A}$ , others are lost, for example Theorem (3.20). But in using the ring  $\mathcal{A}$  we do not lose anything from the Laplace Transformation.

The whole theory of Mikusinski's operators is included in the presented theory, because all his operators are quotient distributions in the sense of Definitions (10.52) and (10.60). But a very important advantage of the presented theory is, that most of quotient distributions can be written in the form of feasible functions  $F(p, t)$  and it is an open problem whether there exist quotient distributions, for which such a representation is not possible. The introduction, for example, of operators, corresponding to our functions  $e^{rt}$  for any real  $r$ , forced Mikusinski to add an additional chapter of his theory. It follows from Theorem (3.20) that, if any of Mikusinski's operators were to be represented in the form  $F(p, t)$ , it would be a feasible function in the sense of Definitions (3.1) and (3.10).

In the theory presented above, differentiating and integrating of functions  $F(p, t)$  is quite natural, because  $p$  is a variable. In Mikusinski's theory it requires a new construction, because  $p$  is an operator.

Example (8.74) shows that there exist problems, which can be solved by use of the theory presented in this paper, but they cannot be solved by the Laplace Transformation or Mikusinski's operators. The function (8.75) is not transformable in the sense of Laplace and its transform (8.78) is not an operator in Mikusinski's sense, because the function

$$e^{-\frac{p^2}{2}}$$

is congruent with zero and the only Mikusinski's operator corresponding to it is the zero operator  $\mathcal{O}$ .

In comparison with both theories, of the Laplace Transformation and of Mikusinski's operators, the theory presented in this paper opens new possibilities for calculating with functions of two variables  $F(p, t)$ . Some such calculations are shown in Chapter VIII. The author is of the opinion that he has not discovered the most important methods in that area yet. For example, Theorem (8.8) enables us to transform any linear differential equation (with any functional coefficients) into a congruence of the form

$$(12.1) \quad F(p, t)x = g(p, t).$$

Generally, such an equation is not easy to be solved analytically. But there are many possibilities for solving it approximately. The author is of the opinion that it could be very interesting to apply methods of that kind to the equation (12.1).

There exist several theories of generalized functions or distributions treating them as sequences of functions  $f_1(t), f_2(t), \dots$ . (See, for example, [6]). The theory presented in this paper could include such theories, because, according to Definitions (2.1) and (3.1), every feasible function  $F(p, t) \in F$  may be treated as a sequence  $F(1, t), F(2, t), \dots$ . However, in such a case we lose the possibility of differentiating and integrating with respect to  $p$ , which reduces significantly the class of methods at our disposal. Therefore we shall not consider such theories in detail.

It remains to consider the difference between the latest theory and the previous one (see [1]). They are not equivalent, because the classes of feasible functions are not equivalent. The latest class includes the previous one, but there exist functions, which are feasible in the latest sense but are not feasible in the previous one. Let us consider, for example, the function

$$(12.2) \quad F(p, t) := e^{2p^2} \cos(e^{p^2} t) .$$

We have for any  $b > 0$

$$\begin{aligned} e^{-bp} (t \times F(p, t)) &= e^{-bp} (1/p \times 1 \times F(p, t)) = e^{-bp} p \int \int e^{2p^2} \cos(e^{p^2} u) du du = \\ &= e^{-bp} p (1 - \cos(e^{p^2} t)) \underset{p \rightarrow \infty}{\Rightarrow} 0 , \end{aligned}$$

which means that  $F(p, t) \in F$ . However, we have

$$\begin{aligned} \int_0^t |e^{-bp} F(p, u)| du &\geq \left| \int_0^t e^{-bp} F(p, u) du \right| = \left| e^{-bp} \int_0^t e^{2p^2} \cos(e^{p^2} u) du \right| = \\ &= \left| e^{-bp} e^{p^2} \sin(e^{p^2} t) \right| \end{aligned}$$

and, for every  $b > 0$ , there exists  $t > 0$  such that for every  $P, Q > 0$  we can find a real  $p > P$  such that

$$\int_0^t |e^{-bp} F(p, u)| du > Q .$$

which means that  $F(p, t)$  is not a feasible function in the sense of the previous theory.

A simpler, but not so important, example is the function

$$(12.3) \quad G(p, t) := e^{pt}, \quad 0 \leq a \leq 1 ,$$

which is also feasible in the sense of the latest theory and is not feasible in the sense of the previous one.

It follows from (4.6) and the fact, that (4.36) would not imply (4.37), that the function

$$(12.4) \quad e^{pt}$$

cannot be feasible and must correspond to  $\infty$ . We see that in the latest theory we can come with (12.3) arbitrarily near to the function (12.4). This means that our class  $F$  of feasible functions is, indeed, constructed in a tight way.

The fact that the class of feasible functions in the latest theory is larger than the corresponding class in the previous one, is not only for the personal satisfaction of the author. It has also very important consequences. For example, in the latest theory it was possible to prove Theorem (4.22), which showed that we had defined the ideal  $Z$  in the optimal way. It was also possible to prove Theorem (3.20), which showed that every quotient distribution equivalent to a function  $F(p, t)$  is a q-distribution. Both theorems (3.20) and (4.22) are fundamental in the latest theory and they do not exist in the previous one.

As it was already said before, there are many open problems connected with the presented theory. Some of them were already mentioned above. Now, let us collect the most important ones.

**Problem 1.** How can equations of the form (12.1) be solved ?

**Problem 2.** Are there any quotient distributions which are not q-distributions ?

**Problem 3.** Does such a function  $g(t) \in L$ ,  $g(t) \neq 0$ , exist, which has transforms, but no transform  $h(p)$  such that  $1/h(p) \in F$  ?

**Problem 4.** Do such functions  $r(t) \in F$  exist, which have no transforms at all ?

**Problem 5.** Is it possible to change Definition (6.1) in such a way that the congruence

$$h(p) = 0$$

would imply

$$h(p) = 0 ?$$

Problem 6. Starting from Theorem (7.35), learn to solve differential equations written with  $\mathbb{D}$ -derivatives of different order, according to Theorem (7.44).

---

The idea used to construct the theory presented in this paper can be applied to construct analogous theories for other integral transformations. The author has obtained already some results in this area, which, however, are not a subject for this paper.

Wollongong, August 1987.

## References

- [1] M.Wormus. A new theory of Operational Calculus.  
Dissertationes Mathematicae LXXX. PWN, Warszawa 1971.
- [2] J.G.Mikusiński. Sur les fondements du calcul opératoire.  
Studio Math. 11 (1950).
- [3] J.G.Mikusiński. Operational Calculus. Pergamon Press, 1959.
- [4] G.Doetsch. Handbuch der Laplace-Transformation.  
Birkhäuser, Basel, Vol.1 (1950), Vol.2 (1955), Vol.3 (1956).
- [5] R.Erdélyi. Operational Calculus and Generalized Functions.  
Holt, Rinehart and Winston, N.York 1966.
- [6] J.Mikusiński and R.Sikorski. The elementary theory of distributions.  
Dissertationes Mathematicae XII. PWN, Warszawa 1957.

d|9|15

<http://rbc.ipipan.waw.pl>