

The Browder Property of Topological Spaces with Abstract
Convexity

Andrzej Wiecek

Institute of Computer Science

Polish Academy of Sciences

P.O. Box 22, 00-901 Warsaw, Poland

Abstract.

The Browder Property of a topological convex set X may be defined as the property that for every relation $\phi \subseteq X \times X$ with nonempty convex sections $\phi(x)$ and open inverse sections $\phi^{-1}(x)$ there exists a "fixed element" x_0 with $(x_0, x_0) \in \phi$.

Actually, Browder was the first author to study, in [1961], the fixed point properties of relations of this kind.

The aim of this paper is to study the Browder property in the general case of convexity given in an abstract axiomatic way. Axiomatic convexity is a rapidly developing field of mathematics; and general references can be found, for instance, in Keimel and Wiegorek's paper [1988] or Wiegorek's [1988].

The first approach to deal with the Browder Property in a generalized convexity setup is due to Komiya [1981]; however, the results presented there, do not go far beyond the usual convexity in linear spaces.

The results in the present paper refer to most general convexity structures; some of them make use of the new concept of compressibility (of a topological space into another, equipped with abstract convexity). The concept itself seems to be very promising; it allows to mimic the usual structure of a linear space but ~~the~~ more detailed studies of this concept itself ^{fall out of} ~~fall out of~~ the scope of this paper. Perhaps I should only add that this concept is also relevant when dealing with the Kakutani Property and

Selections of Relations; this subject ^(should) will be developed
in separate publications. ^{(for instance,}

The Browder Property is very important, when dealing
with the existence of ^{Nash} equilibria in games with constraints.
Certain techniques, ^{of proof} introduced by Gale and Mes-Cotell
in [] and by Borglin and Keidling in [] and
developed by Vanuvelis and Preblecher in []
derive the existence of a Nash equilibrium from,
among others, the Browder Property of some spaces
related to players' strategy sets. Currently, in game
theory there arises a need to deal with players'
strategy sets equipped with convexities no different
from the usual and, consequently, there ^{arises} a need
for theorems on the Browder Property of spaces
equipped with generalized convexity. Applications of
this kind will be presented separately by Więciorak
in [forthcoming].

1. General definitions

A convexity on a set X is a family \mathcal{C} of subsets of X such that $X \in \mathcal{C}$ and $\cap \mathcal{C}' \in \mathcal{C}$ for any $\mathcal{C}' \subseteq \mathcal{C}$. Elements of \mathcal{C} are called convex sets.

For any set $A \subseteq X$ we define the hull of A as $\cap \{C \in \mathcal{C} \mid A \in C\}$; we denote it by $\text{hull } A$.

We say that a topological space X has the Browder Property (w.r.t. a fixed convexity \mathcal{C} on X) whenever, for every $\phi \subseteq X \times X$ with nonempty convex sections $\phi(x)$ and open inverse sections $\phi^{-1}(x)$ there exists x_0 such that $x_0 \in \phi(x_0)$.

(The) Browder Property obtained via Selection Techniques

3. The Browder Property of Polytopes and of the Whole Space

Given a convexity \mathcal{C} on a set X , a $\overset{\text{convex}}{\text{polytope}}$ in X is a hull of a finite set.

For any $A \subseteq X$ the family $\{A \cap C \mid C \in \mathcal{C}\}$ is a convexity on A ; we call it restricted convexity.

Proposition. Let X be a nonempty compact space equipped with a convexity \mathcal{C} , and let ϕ have nonempty sections $\phi(x)$ and open inverse sections $\phi^{-1}(x)$. There exists a nonempty convex polytope $P \subseteq X$ such that also $\phi|P$ ($= \phi \cap P^2$) has nonempty sections.

Proof: The family $\{\phi^{-1}(x) \mid x \in X\}$ is an open cover of X ; by compactness, there exists a finite subcover $\{\phi^{-1}(x_1), \dots, \phi^{-1}(x_n)\}$. It suffices to take $P := \text{hull}\{x_1, \dots, x_n\}$.

Theorem. A nonempty compact space X , equipped with some ^(a) convexity \mathcal{C} has the Browder property whenever its all nonempty convex polytopes have the Browder property.

In the Theorem below the polytopes are regarded as topological spaces with the trace topology and their Browder property is understood w.r.t. the restricted convexity.

Proof: Let $\phi \subseteq X \times X$ have nonempty convex sections $\phi(x)$ and open inverse sections $\phi^{-1}(x)$

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