

# Existence of equilibria in normal games with constraints

Andrzej Wieciorak  
Institute of Computer Science  
Polish Academy of Sciences  
P.O. Box 22, 00-901 Warsaw

Abstract. The paper presents general theorems on the existence of equilibria in normal games with constraints, also known as abstract economies. In particular, we deal with: 1. players' preferences given in the form of preference relations, pseudo-utility or utility functions, and 2. players' strategy sets equipped with various special structures (such as convexity) binding the constraints and preferences strongly enough to guarantee the existence of an equilibrium.

AMS Subject Classification: 90D10 (52A01, 54H25, 90A10)

Key words: equilibrium, game with constraints, abstract economy, generalized convexity

Abbreviated title: Equilibria in normal games

The paper deals with normal games with arbitrarily many players. Apparently, <sup>the</sup> players choose their strategies independently, but, given a pre-assigned system of players' strategies  $s$ , the ~~players to the~~ ~~binding~~ constraints force player  $\xi$  to choose his strategy in a set  $\Gamma_\xi(s)$ . Players' preferences allow them to compare various systems of strategies  $s$ ; an equilibrium is a system of strategies  $s^*$  such that  $s_\xi^* \in \Gamma_\xi(s^*)$  holds for each player  $\xi$  who, moreover, has no option ~~in~~  $\Gamma_\xi(s^*)$  ~~leading him to~~ to a system of strategies preferred by him to  $s^*$ .

The above scheme generalizes the usual Nash [1951] equilibrium scheme; It has been first considered by Debreu in [1952]; relevant contributions are due to Shafer and Sonnenschein [1975], Gale and Mas-Colell [1975], Prakash and Sertel [1974] (cf. Ichiishi [1983], p. 70). Some authors have used the designation "abstract economy" ~~implying an economic interpretation~~ ~~and~~ ~~not~~ ~~an~~ ~~economic~~ ~~interpretation~~ while ~~the~~ ~~name~~ ~~game~~ of the object  $s$  (other authors prefer to talk about "normal games with constraints"); in the present paper we abbreviate the latter to "game".

The sole objective of the present paper is in the existence of equilibria. We assume that players' strategy sets are topological spaces additionally equipped with some special structures, such as convexity, binding players' constraints and preferences in such a way as to guarantee the existence of fixed points of appropriately constructed relations. The latest is derived from the author's new results concerning the Kakutani <sup>of products</sup> property (Wieczorek [1989]) as well as from classical theorems such as the Eilenberg-Montgomery [1946] Fixed Point Theorem. As usual, a fixed point:

occurs to be an equilibrium in the game.

Players' preferences can be given in the form of preference relations (e.g. Shaffer and Sonnenschein [1975], Gale and Max-Colell [1975], utility functions - e.g. Debreu [1952] or, as we call them here, pseudo-utility functions - e.g. Ichiishi [1983], p. 70, or Wiczkorek [1985]). Obviously, the preference relations, even restricted, for player  $\xi$ , to the set  $\{(s, s') \mid s_\xi = s'_\xi\}$ , furnish a sufficient conceptual framework to define and study the equilibria but, on the other hand, preferences given in the form of utility or pseudo-utility functions may occur specially "nice" to handle (in the sense that their very natural properties imply the desired results).

## I. Notation and definitions

Throughout the paper we shall use the following notation:  
given sets  $X, Y$ , a relation  $\phi \subseteq X \times Y$  and an  $x \in X$ ,  $\phi(x)$  denotes  
 $\{y \in Y | (x, y) \in \phi\}$ ;

a fixed point of a relation  $\phi \subseteq X \times X$  is any  $x \in X$  such that  
 $x \in \phi(x)$ ;

given a function  $s$ ,  $\xi \in \text{Dom } s$  and any  $\sigma$ ,  $(s^{-\xi}; \sigma)$  denotes  
a function  $s'$  such that  $\text{Dom } s' = \text{Dom } s$ ,  $s'_\xi = s'_\xi$  for all  
 $\zeta \in \text{Dom } s$ ,  $\zeta \neq \xi$ , while  $s'_\xi = \sigma$ .

The product of topological spaces is also regarded as a  
topological space with its product (Tychonoff) topology.

Recall that, given topological spaces  $X$  and  $Y$ , a relation  
 $\phi \subseteq X \times Y$  is upper semicontinuous (abbreviated u.s.c.) if its sections  $\phi(x)$  are closed and the following condition holds:

for all  $x_0 \in X$  and every neighborhood  $U$  of  $\phi(x_0)$  there is a neighborhood  $V$  of  $x_0$  such that  $\phi(x) \subseteq U$  for all  $x \in V$ .

We say that a topological space  $X$  has the Kuratowski pro-  
perty w.r.t. a family of sets  $\mathcal{K} \subseteq 2^X$  whenever, for every  
u.s.c. relation  $\phi \subseteq X \times X$  such that  $\phi(x) \in \mathcal{K} \setminus \{\emptyset\}$  for all  
 $x \in X$ , there exists  $x_0$  such that  $x_0 \in \phi(x_0)$ .

## II Definitions Pre-games and games

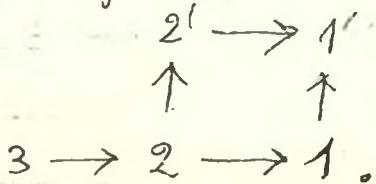
A pre-game consists of two indexed families of nonempty sets,  $(S_\xi \mid \xi \in \Xi)$  and  $(\Gamma_\xi \mid \xi \in \Xi)$ , such that (we denote ad hoc  $\underline{S} := \prod(S_\xi \mid \xi \in \Xi)$ ) for every  $\xi \in \Xi$ ,  $\Gamma_\xi \subseteq \underline{S} \times S_\xi$  and, for every  $\underline{s} \in \underline{S}$ ,  $\Gamma_\xi(\underline{s}) \neq \emptyset$ .

Elements of  $\Xi$  are players in the pre-game, elements of  $S_\xi$  are player's  $\xi$  strategies while  $\Gamma_\xi$  is his constraining relation.

Here are the most common ways to express player's  $\xi$  preferences in a pre-game:

1. a preference relation  $P_\xi \subset \underline{S} \times \underline{S}$ , such that  $(\underline{s}, \underline{s}) \in P_\xi$  holds for no  $\underline{s} \in \underline{S}$ ;
- 1'. a limited preference relation  $P_\xi \subset \underline{S} \times S_\xi$ , such that  $(\underline{s}, s_\xi) \in P_\xi$  holds for no  $\underline{s} \in \underline{S}$ ;
2. a pseudo-utility function  $p_\xi: \underline{S} \times \underline{S} \rightarrow \mathbb{R}$  such that  $p_\xi(\underline{s}, \underline{s}) \leq 0$  holds for all  $\underline{s} \in \underline{S}$ ;
- 2'. a limited pseudo-utility function, real-valued with  $\underline{S} \times S_\xi \supseteq \text{Dom } p_\xi \supseteq \Gamma_\xi$ , such that  $p_\xi(\underline{s}, s_\xi) \leq 0$  holds for all  $\underline{s} \in \underline{S}$ ;
3. a utility function  $u_\xi: \underline{S} \rightarrow \mathbb{R}$ .

Some types of preferences generate some others, according to the following diagram:



The generation is determined by the following formulae:

1' → 1'. given  $P_S$  define

$$P_S := \{(s, \sigma) \in S \times S_\xi \mid ((s^\xi; \sigma), s) \in P_\xi\};$$

2' → 2'. given  $P_S$  define, for  $s \in S$  and  $\sigma \in S_\xi$ ,

$$p_\xi(s, \sigma) := P_\xi((s^\xi; \sigma), s);$$

2' → 1. given  $p_\xi$  define

$$P_\xi := \{(s, s') \in S \times S \mid p_\xi(s, s') > 0\};$$

2' → 1'. given  $p_\xi$  define

$$P_\xi := \{(s, \sigma) \in S \times S_\xi \mid p_\xi(s, \sigma) > 0\};$$

3' → 2. given  $u_\xi$  define, for  $s, s' \in S$ ,

$$f_\xi(s, s') := u_\xi(s) - u_\xi(s').$$

A game could be defined as a system consisting of a pre-game and a specification of all players' preferences which, generally, might be of either type. However, for simplicity of notation, we restrict the definition to the case of the most economical preferences of type (1').

A normal game with constraints, shortly game, is a system consisting of a pre-game  $\gamma = (S_\xi, \Gamma_\xi \mid \xi \in \Xi)$  and an indexed family of limited preference relations  $(P_\xi \mid \xi \in \Xi)$ . An

~~strategy profile satisfying all constraints~~  $\overrightarrow{\text{equilibrium}}$  for  $\gamma$  is a system of strategies  $s = (s_\xi \mid \xi \in \Xi)$  such that for every  $\xi \in \Xi$ ,  $(s, s_\xi) \in \Gamma_\xi$  and  $\Gamma_\xi(s) \cap P_\xi(s) = \emptyset$ .

~~Remark. Let  $\gamma$  be a game whose elements are sets of players in an fixed game. Then an  $s$  is a  $\gamma$ -equilibrium if and only if it is an  $\alpha$ -equilibrium for every  $\alpha \in \gamma$ .~~

### III. General remarks

In this section we shall study the most elementary properties of games implying the existence of equilibria. The central result in this section and in the whole paper is Theorem (6), trivial but extremely fruitful in consequences.

Let  $\xi$  be a player in a pre-game  $(S_\xi, \Gamma_\xi | \xi \in \Xi)$ . We define a relation  $e_\xi \subseteq S \times S_\xi$  letting, for  $s \in S$ ,

$$\begin{aligned} e_\xi(s) &:= \Gamma_\xi(s) \text{ if } P_\xi(s) = \emptyset, \\ &:= S_\xi \setminus \{s_\xi\} \text{ if } P_\xi(s) \neq \emptyset. \end{aligned}$$

A relation  $E \subseteq S \times S$  is defined by

$$E(s) := \prod_{\xi \in \Xi} (e_\xi(s) | \xi \in \Xi),$$

for  $s \in S$ .

The following observation is obvious:

5. Proposition. Let  $\gamma = (S_\xi, \Gamma_\xi, P_\xi | \xi \in \Xi)$  be a game and let  $s \in S$ . The following conditions are equivalent:

- a.  $s$  is an equilibrium for  $\gamma$ ;
- b.  $s$  is a fixed point for  $E$ ;
- c.  $s$  is a fixed point for a relation  $E' \subseteq E$ .

Suppose that, given a pre-game  $(S_\xi, \Gamma_\xi | \xi \in \Xi)$ , all  $S_\xi$  are topological spaces; let  $P_\xi$  be a limited preference relation of a player  $\xi \in \Xi$ . We shall say that a family of sets  $K_\xi \subseteq 2^S$  is adjusted (to the player  $\xi$ ) if there exists an upper semi-continuous relation  $e \subseteq e_\xi$  such that  $e(s) \in K_\xi \setminus \{\emptyset\}$  holds for all  $s \in S$ .

6. Theorem. let  $\gamma = (S_\xi, \Gamma_\xi, P_\xi | \xi \in \Xi)$  be a game and assume that all  $S_\xi$  are topological spaces. If, for every player  $\xi$ , there is an adjusted family of sets  $K_\xi \subseteq 2^{S_\xi}$  while the product  $\prod(S_\xi | \xi \in \Xi)$  has the Kakutani property w.r.t. the family of all boxes  $\prod(K_\xi | \xi \in \Xi)$ , with  $K_\xi \in K_\xi$  for all  $\xi \in \Xi$ , then there exists an equilibrium for  $\gamma$ .

In order to verify Theorem (6) one only has to recall the definition of the Kakutani property and to compare it with the remaining notions appearing in its formulation.

#### IV. Abstract convexities

Throughout the paper we shall be working in an abstract convexity framework. Here are the basic definitions along with descriptions of the most important special cases which are covered by the theory developed in this paper.

A convexity on a topological space  $S$  is a family  $\mathcal{K}$  of closed subsets of  $S$  which contains  $S$  as an element and which is closed under arbitrary intersections. Elements of  $\mathcal{K}$  are called closed convex sets (there might be subsets of  $S$  not in  $\mathcal{K}$  also interpreted as convex sets). A convexity  $\mathcal{K}$  is normal if for every  $K \in \mathcal{K}$  and every closed set  $F$  disjoint from  $K$  there exists  $K' \in \mathcal{K}$  also disjoint from  $F$ , such that  $K \subseteq \text{Int } K'$ .

Given an indexed family of topological spaces  $(S_\xi \mid \xi \in \Xi)$  and an indexed family of convexities  $(\mathcal{K}_\xi \mid \xi \in \Xi)$ , with

$\mathcal{K}_\xi$  being a convexity on  $S_\xi$ , the box convexity on the product space  $\prod(S_\xi \mid \xi \in \Xi)$  is the convexity consisting of all the products  $\prod(K_\xi \mid \xi \in \Xi)$  such that  $K_\xi \in \mathcal{K}_\xi$  for every  $\xi \in \Xi$ .

We say that a convexity  $\mathcal{K}$  on a space  $S$  is appropriate. (in order to avoid too many technical terms we slightly modify here the original terminology used by Wieczorek in [1989]) if there exists a finitely multiplicative family  $\mathcal{G} \subseteq 2^S$  which is a base of the topology in  $S$  (i.e. for

every open set  $O \subseteq S$  and  $x \in O$  there exists  $G \in \mathcal{G}$  such that  $x \in \text{Int } G \subseteq G \subseteq O$ , satisfying <sup>the</sup> conditions:

- i. for every  $A \in K$  and  $G, G' \in \mathcal{G}$  such that  $A \cap G \cap G' = \emptyset$  there exist  $K, K' \in K$  such that  $A \cap G \subseteq K \setminus K'$ ,  $A \cap G' \subseteq K' \setminus K$  and  $K \cup K' = A$ ;
- ii. for every  $K, K' \in K$  such that  $K \cup K' \in K$  and every  $G \in \mathcal{G}$ , the conditions  $G \cap K \neq \emptyset$  and  $G \cap K' \neq \emptyset$  imply  $G \cap K \cap K' \neq \emptyset$ .

Given a convexity  $\mathcal{K}$  on a topological space  $X$  write, for a set  $A \subseteq X$ ,  $\text{hull } A := \{K \in \mathcal{K} / A \subseteq K\}$ . Say that  $\mathcal{K}$  has the compact hull property whenever  $\text{hull}(K \cup K')$  is compact for any compact convex sets  $K, K'$ . Say that  $\mathcal{K}$  is compact-normal if for every compact set  $C \in \mathcal{K}$  and every closed set  $F$  disjoint from  $C$  there exists a compact set  $D \in \mathcal{K}$ , also disjoint from  $F$  such that  $C \subseteq \text{Int } D$ . Obviously, for a compact space  $X$  the concepts of normality and compact-normality coincide.

A set  $A \subseteq X$  is convex-like if  $\text{hull}(K \cup K') \subseteq A$  for any compact sets  $K, K' \in \mathcal{K}$ .

Finally, we say that a continuous function  $f: X \rightarrow \mathbb{R}$  is quasi-concave (w.r.t.  $\mathcal{K}$ ) whenever  $\{x / f(x) \geq \alpha\} \in \mathcal{K} \cup \{\emptyset\}$  for every real  $\alpha$ .

Given topological spaces  $U, X$  and a convexity  $\mathcal{K}$  on  $X$  we refer to a set  $B \subseteq U \times X$  as a block (w.r.t.  $U, X$  and  $\mathcal{K}$ ) whenever it is a product of a compact set in  $U$  and a compact element of  $\mathcal{K}$ .

We shall now list three very important special cases of admissible convexities (more details can be found in Wiczkorek [1]):

8. the family of all closed and convex (in the usual sense) subsets of a compact convex set in a locally convex Hausdorff topological vector space is a normal and ~~appropriate~~ convexity;
9. a tree is a compact connected and locally connected space  $X$  in which every two distinct points  $x$  and  $x'$  can be separated by some  $y$  (in the sense that  $x$  and  $x'$  belong to different connected components of  $X \setminus \{y\}$ ); the convexity on  $X$  consisting of all closed connected subsets of  $X$  is both normal and ~~appropriate~~ appropriate;
10. a connecting function on a topological space  $X$  is a continuous function  $c: [0; 1] \times X^2 \rightarrow X$  such that

$$c(\alpha, x, x') = c(1, x, x') = c(0, x', x) = x$$

holds for all  $\alpha \in [0; 1]$  and all  $x, x' \in X$ . The convexity generated by  $c$  consists of all convex sets  $A \subseteq X$  such that for all  $\alpha \in [0; 1]$  and  $x, x' \in X$ ,  $x, x' \in A$  implies  $c(\alpha, x, x') \in A$ . If this convexity is normal and satisfies the condition:

for every  $x, x', x_1, x'_1 \in X$ , every  $\alpha \in (0; 1)$  and  $\beta, \beta' \in (0; 1)$  there exists  $r \in [0; 1]$  such that  $c(r, c(\beta, x, x_i)), c(\beta', x'_1, x'_1)$  belongs to every closed convex set including  $c(\alpha, x, x')$ ,  $x_1$  and  $x'_1$  then it is ~~appropriate~~. In turn, the convexity generated by  $c$  is normal, for instance, in the case where  $X$  is metrizable with a metric  $g$  satisfying the condition:

for every  $\varepsilon > 0$ ,  $x, x', x_1, x'_1 \in X$  such that  $g(x, x') \leq \varepsilon$  and  $g(x_1, x'_1) \leq \varepsilon$  and every  $\alpha \in [0; 1]$  there exists  $\beta \in [0; 1]$  such that  $r(c(\alpha, x, x_1), c(\beta, x'_1, x'_1)) \leq \varepsilon$ .

## V. The Kakutani property of products

Here ~~there~~ are ~~several~~ two <sup>important</sup> fixed point theorems which can be applied when proving the existence of equilibria in games. The first of them, Theorem (6), can be formulated directly while the other requires several auxiliary steps.

7. Theorem. For  $\xi \in \Xi$ , let  $S_\xi$  be a compact metric acyclic ANR (absolute neighborhood retract) and let  $K_\xi$  be the family of all acyclic subsets of  $S_\xi$ . The space  $\prod(S_\xi | \xi \in \Xi)$  has the Kakutani property w.r.t. the family of all boxes  $\prod(K_\xi | \xi \in \Xi)$ , with  $K_\xi \in K_\xi$  for all  $\xi \in \Xi$ .

Theorem (7) follows, in the case of finite  $\Xi$ , immediately from the Eilenberg-Montgomery [1946] Fixed Point Theorem. The general case follows from the observation that the Kakutani property of the product of finitely many factors implies the Kakutani property of the product of arbitrarily many factors (see Wierzbek [1989], Theorem (13)).

~~The following~~ result has been proved by Wierzbek in [Another]

8. Theorem. For  $\xi \in \Xi$ , let  $K_\xi$  be a normal and ~~appropriate~~ convexity on a nonempty compact space  $S_\xi$ . The space  $\prod(S_\xi | \xi \in \Xi)$  has the Kakutani property w.r.t. the box convexity

## VI. Adjusted convexities

in the case of preferences generated by pseudo-utility functions

The basic fact in this section is included in the following theorem:

12. Theorem. Let  $(S_\xi, \Gamma_\xi | \xi \in \Xi)$  be a pre-game and assume that all  $S_\xi$  are topological spaces<sup>and</sup> let a player's limited preference relation  $P_\xi$  be generated by a limited pseudo-utility function  $p_\xi$ . A convexity<sup>(\*)</sup> on  $S_\xi$  ~~is adjusted~~ whenever:

- a. all  $S_\xi$  are compact;
- b.  $\Gamma_\xi$  is upper and lower semi-continuous;
- c. for every  $s \in S$ ,  $\Gamma_\xi(s) \in K_\xi$ ;
- d.  $p_\xi$  is continuous; and
- e. for every  $s \in S$ ,  $p_\xi(s, \sigma)$  is quasi-concave in  $\sigma$ .

Proof: Define a relation  $F_\xi \subseteq S \times S_\xi$  letting, for  $s \in S$ ,

$$F_\xi(s) := \{\sigma \in \Gamma_\xi(s) | p_\xi(s, \sigma) = \max \{p_\xi(s, \tau) | \tau \in \Gamma_\xi(s)\}\}.$$

By a known ~~theorem~~ theorem (see e.g. Wu [1961], p. 397),  $F_\xi$  is upper semi-continuous and, by (c) and (e),  $F_\xi(s) \in K_\xi$  for every  $s \in S$ .

One can easily see that, in the formulation of Theorem (12), it suffices to assume, instead of (c) and (e), the following:  
e'. for every  $s \in S$  and real  $\alpha$ ,  $\Gamma_\xi(s) \cap \{\sigma | p_\xi(s, \sigma) \geq \alpha\} \in K_\xi$ .

Notice that conditions (12. c-d) are clearly satisfied whenever  $p_{\underline{s}}$  is generated by a continuous utility function  $u_{\underline{s}}(\underline{s})$ , quasi-concave in  $\underline{s}_{\underline{s}}$ .

Now there arises a natural question, when a limited preference relation  $P_{\underline{s}}$  is generated by a limited pseudo-utility function satisfying (12. c-d). An answer is given by the following Theorem (13) which is an immediate consequence of Theorem proved by Wiczkorek in [1988]:

B

13. Theorem. Let  $(S_\xi, \Gamma_\xi \mid \xi \in \Xi)$  be a pre-game in which all  $S_\xi$  are topological spaces and let  $\xi \in \Xi$  be fixed. Let  $K_\xi$  be a convexity on  $S_\xi$  and let  $P_\xi$  be a limited preference relation of player  $\xi$ . If

- a. all  $S_\xi$  are locally compact ~~while~~ all but finitely many are compact;
  - b.  $K_\xi$  is compact-normal and has the compact hull property;
  - c.  $P_\xi$  is open and it can be represented as a union of countably many blocks;
  - d. for every  $s \in S$ ,  $P_\xi(s) \cap \Gamma_\xi$  is convex-like (w.r.t.  $K_\xi$ );
- then  $P_\xi$  is generated by a continuous limited pseudo-utility function  $f_\xi$  such  $P_\xi(s, \xi)$ , quasi-concave in  $s$ .

~~Theorem (12) and (13) immediately imply~~

The assumption (c) in Theorem (13) is equivalent to c';  $P_\xi$  is open; under the following additional assumptions: the spaces  $T := \prod(S_\xi \mid \xi \neq \xi)$  and  $S_\xi$  are perfectly normal with either  $T$  or  $S_\xi$  being metrizable,  $T$  is  $\sigma$ -compact and e.  $S_\xi$  has a base composed of countably many elements of  $K_\xi$ . For instance, this is the case if  $\Xi$  is countable, all  $S_\xi$ ,  $\xi \neq \xi$ , are compact and metrizable while  $S_\xi$  is perfectly normal (e.g. metrizable) and it satisfies (e).

~~and (13)~~  
The Theorems (12) and (13) immediately imply:

14. Corollary. Let  $(S_\xi \mid \xi \in \Xi)$  be a pre-game in which all  $S_\xi$  are topological spaces and let  $P_\xi$  be a player's limited preference relation. A convexity  $K_\xi$  on  $S_\xi$  is adjusted whenever:

- a. all  $S_\xi$  are compact;
- b.  $\Gamma_\xi$  is upper and lower semi-continuous;
- c.  $\Gamma_\xi(\underline{s}) \in K_\xi$  for every  $\underline{s} \in \underline{S}$ ;
- d.  $K_\xi$  is compact-normal and has the compact hull property;
- e.  $P_\xi$  is open and it can be represented as a union of countably many blocks; and
- f. for every  $\underline{s} \in \underline{S}$ ,  $P_\xi(\underline{s})$  is convex-like (w.r.t.  $K_\xi$ ).

15. Theorem. Let  $(S_\xi \mid \xi \in \Xi)$  be a family of compact spaces and let  $\xi \in \Xi$ . Let  $K_\xi \subseteq \underline{S} \times S_\xi$  and  $P_\xi \subseteq \underline{S} \times S_\xi$  and let  $K_\xi$  be an appropriate [local] convexity on  $S_\xi$ . If

- a.  $S_\xi$  is metrizable;
  - b.  $\Gamma_\xi$  is upper semi-continuous;
  - c.  $\Gamma_\xi \cap P_\xi$  is lower hemi-continuous; and
  - d.  $\Gamma_\xi(\underline{s}) \cap P_\xi(\underline{s}) \in D(K)$ , for all  $\underline{s} \in \underline{S}$  while  $\Gamma_\xi(\underline{s}) \in K_\xi$  whenever  $P_\xi(\underline{s}) = \emptyset$ ;
- then  $K_\xi$  is adjusted to  $\xi$ .

VII. ~~Adjusted convexities from the envelope of continuous selections~~  
~~Adjusted convexities obtained via continuous~~  
selections

The topic of the existence of continuous selections for lower semi-continuous relations, in the case of abstract convexities, is still ~~not considered in the literature~~. This requires systematic studies. Consequently we are ~~not yet~~ able to formulate ~~yet~~ the most general results concerning the ~~structure of the problem of~~ adjusted convexities.

However, an analysis of the existing results among them the classical papers by Michael, [1956], [1956], [1957], [1959] and van de Vel's report [1982] allows to formulate the following result, Theorem (16). We must add that this result is not explicitly stated in either of these papers. It still requires a proof, probably including many elements of the proofs in the above papers.

We begin with a few necessary definitions.

Given a relation  $\phi \subseteq X \times X$ , a selection for  $\phi$  is any function  $f: \{x \mid \phi(x) \neq \emptyset\} \rightarrow X$  such that  $f(x) \in \phi(x)$  whenever  $\phi(x) \neq \emptyset$ .

If  $X$  and  $Y$  are topological spaces then a relation  $\phi \subseteq X \times Y$  is said to be lower semi-continuous (l.s.c.) if and only if for every set  $A \subseteq X$  and  $x_0 \in \bar{A}$ ,  $\phi(x_0)$  is included in the closure of  $\cup \{\phi(x) \mid x \in A\}$ .

Given a convexity  $K$  on a topological space  $X$  and a set  $A \subseteq X$ , a set  $B \subseteq A$  is extreme in  $A$  whenever  $B' \cap B = \emptyset$  for every  $B' \subseteq A \setminus B$ . For any set  $A \subseteq X$  we denote by  $J_A$  the set of all elements of  $A$  which do not belong to any proper extreme of  $A$  (cf. Wiczkorek [1989]).

Let  $\mathcal{D}(K)$  denote the family of all sets  $A \subseteq X$  which are unions of directed families of elements of  $K$  and which have the following property such that  $A \supseteq T(\bar{A})$ .

16. Theorem. Let  $U$  and  $X$  be compact spaces and let  $K$  be an appropriate [and local?] convexity on  $X$ . Let  $\Phi \subseteq U \times X$  be lower semi-continuous and let a.  $X$  be separable and  $\Phi(u) \in \mathcal{D}$  for all  $u \in U$ ; or b.  $\Phi(u) \in K$  for all  $u \in U$ .  
Then  $\Phi$  admits a continuous selection.

This is not finished, Section VII will be rewritten and 3 more sections, probably:

VIII. Browder type theorems

IX. Other results concerning adjusted convexities

X. Conclusions,  
will be added.

References

- K. Border, 1985, Fixed point theorems with applications to economics and game theory, Cambridge University Press, Cambridge
- A. Borglin and H. Keiding, 1976, Existence of equilibrium actions and of equilibrium: A note on the "new" existence theorems, J. Math. Econ. 3, 313-316
- G. Debreu, 1952, A social equilibrium existence theorem, Proc. Nat. Acad. Sci. U.S.A. 38, 886-893
- D. Gale and A. Mas-Colell, 1975, An equilibrium existence theorem for a general model without ordered preferences, J. Math. Econ. 2, 9-15
- Ky Fan, 1952, Fixed point and minimax theorems in locally convex topological linear spaces, Proc. Nat. Acad. Sci. U.S.A. 38, 121-126
- I.L. Glicksberg, 1952, A further generalization of the Kakutani fixed point theorem with application to Nash equilibrium points, Proc. Amer. Math. Soc. 3, 170-174
- T. Ichiishi, 1983, Game theory for economic analysis, Academic Press, New York

- A. Mas-Colell, 1986, The price equilibrium existence problem in topological vector lattices, *Econometrica* 54, 1039 - 1053
- J. F. Nash, 1950, Equilibrium points in n-person game, *Proc. Nat. Acad. Sci. U.S.A.* 36, 48-49
- P. Prakash and M. R. Sertel, 1974, On the existence of noncooperative equilibria in social systems, Discussion Paper 92, Center for Mathematical Studies in Economics and Management Science, Northwestern, Evanston
- W. Shafer and H. Sonnenschein, 1975, Equilibrium in abstract economies without ordered preferences, *J. Math. Econ.* 2, 345 - 348
- M. van de Vel, 1982, A selection theorem for topological convex structures, Report 212, Vrije Universiteit Amsterdam, Subfaculteit Wiskunde en Informatica
- A. Wieczorek, 1986, Constrained and Indefinite Games and Their Applications, *Dissertationes Math.* 246, 1-43
- A. Wieczorek, 1987, Parametrized pseudo-utility representations, Report 611, Institute of Computer Science, Polish Academy of Sciences, Warsaw

A. Wiczkorek, 1989, Spot functions and peripherals: Krein-Milman type theorems in abstract setting, J. Math. Anal. Appl. 136, to appear

A. Wiczkorek, 1989, The Kakutani property and the fixed point property of topological spaces with abstract convexity, Report 6., Institute of Computer Science, Polish Academy of Sciences, Warsaw

Wu Wen-tsün, 1961, On non-cooperative games with restricted domains of activities, Scientia Sinica 10, 387-409

N. Yannelis and N. Prabhakar, 1983, Existence of maximal elements and equilibria in linear topological spaces, J. Math. Econ. 12, 233-245

K. Keimel and A. Wiczkorek, 1988, Kakutani property of the polytopes implies the Kakutani property of the whole space, J. Math. Anal. Appl. 130, 97-109

E. Michael, 1956, Continuous selections I, Ann. Math. 63, 361-382  
1956, Continuous selections II, Ann. Math. 64, 562-580  
1957, Continuous selections III, Ann. Math. 65, 375-390

E. Michael, 1959, Convex structures and continuous selections, Can. J. Math. 11, 556-575