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Axiomatic Convexity and Its Applications

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1. There are lots of results in mathematics dealing with convex sets in which the meaning of convexity is dubious in the sense that there is no immediate reason why just convex sets should be involved. Examples: begin with Brouwer or Schauder theorem but go to Kakutani or Michael's Selection theorems; however there are also hundreds of other theorems, sometimes also involving topology in which arises a question: why just ~~topology~~ convex sets?
An analysis proves that - practically in all instances - what really matters are just set-theoretic relations between convex sets or closed convex, open convex sets in situations involving topology.
Examples are theorems involving dimension of a space number of convex sets which may intersect provided some their subfamilies intersect (etc. etc.)

2. The idea of replacing convex sets by some other families of sets is therefore natural ~~and~~ though it is not new. Let me now briefly review concepts generalizing the usual convexity

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3. The following seems very general and still very reasonable:
a convexity on a set X is a family $\mathcal{C} \subseteq 2^X$ with $X \in \mathcal{C}$ which is absolutely multiplicative. The definition was first given by Levi [1951] and then re-~~4y~~ discovered and the theory developped in '70 by people like ~~van de~~ Jamison, Van de Vel and many many others.

4. Of course, convexity is a generalization of topology in a set (w.r.t. closed sets) but one should not think this way: intuitions are entirely different

5. Given convexity $\mathcal{C}$ one can define operation of hull (convex hull), $\text{hull}\, A := \bigcap\{C \in \mathcal{C} \mid A \subseteq C\}$
Conversely one can begin at with abstract oper. of hull: $2^X \to 2^X$ satisfying $A \subseteq \text{hull}\, A = \text{hull}\,\text{hull}\, A$
This is the Moore operation (beginning of $\overline{XX}$) more general than Kuratowski's operation of closure
In turn the Moore operation generates a convexity:
A is convex if and only if $A = \text{hull}\, A$
The two approaches are just equivalent.

6. One may assume but not necessarily so that $\emptyset \in \mathcal{C}$, and union of an upward directed family of convex sets is also convex; in this case the thing is called alignment

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7. An important special case: X is equipped with a relation of being "between" $B \subseteq X \times X \times X$, $xyz \in B$ means "y is between x and z" typically $xyz \in B \Rightarrow zyx \in B$; $xxy \in B$; not necessarily $xyz \in B$ $xy'z \in B$ $yty' \in B \Rightarrow xtz \in B$

Then ~~let~~ A is <u>convex</u> if $x, x' \in A$, $xyx' \in B \Rightarrow y \in A$. <u>This conv. is an alignment</u>

A special case of this:

there is a function $c: [0,1] \times X^2 \to X$ s.t. $c(0, x, x') =$ $= c(1, x', x) = c(\alpha, x', x')$; not necessarily $c(\alpha, x, x') =$ $c(1-\alpha, x', x)$.

A function like this generates a relation of being between, y is between x & x' provided $y = c(\alpha, x, x'$ for some α.

8. Quite ~~often~~ similar things are considered for not 2 but n factors, relation of being between n-elements or funcions $c_n : \Delta_{n-1} \times X^n \to X$ The two approaches are not quite equivalent (we are not going to details.

9. Sometimes also - initially given families of sets the way Ky Fan did in 1963.

10. Generally - <u>see the Book of Soltan</u>

This is a meaning of convexity!!!

11. In case of topology - it is interesting to consider just only a family of closed sets interpreted as convex

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12. In the topological case we have some interesting "seperation" conditions which may be either taken as primitive or derived from the others (such papers do exist!)

<u>Hahn Banach ~~type~~ property</u>: given closed convex disjoint sets C, C' there exist closed convex sets D, D' with $D \cup D' = X$, $C \subseteq D \setminus D'$, $C' \subseteq D' \setminus D$; you may assume: compact

mimic topology T_i-spaces

<u>regularity</u>: closed convex C and $x \notin C$; there exists closed convex D with $x \notin D$, $C \subseteq \operatorname{Int} D$

<u>normality</u>: closed ~~E~~ convex disj. sets C, C'; there exist closed convex disj sets D, D' with $C \subseteq \operatorname{Int} D$, $C' \subseteq \operatorname{Int} D'$

<u>compact normality</u>: normality restricted to compact sets only

13. Real life examples:

A. Of course: usual convexity in ~~fa~~ (convex subset of a) topological vector space

B. X is a locally connected tree-like space (i.e. any ~~R~~ two elements can be seperated by third one) "convex" sets are connected sets. If it is also compact then it is normal.

C. ~~Com~~ Topological spaces admitting complete metric ~~admitting~~ strongly convex ~~metric space~~ in the sense of Borsuk, i.e. $\forall_{x, x'}$ $\exists$ ex. one y s.t. $d(x,y) = d(y,x') = \frac{1}{2} d(x,x')$.

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compact hull property = given compact convex sets C, C', hull $(C \cup C')$ is compact.
set G is convex-like if $K \cup K' \subseteq G$, K, K' compact convex sets then hull $(K \cup K') \subseteq G$
quasi-concave.

Convexity is <u>penetrating</u> if for all $K, K', C \in \mathcal{C}$ ($K \cup K' \in \mathcal{C}$ $C \cap K \neq \emptyset$ and $C \cap K' \neq \emptyset$) implies $C \cap K \cap K'$.
Convexity is <u>local</u> if the space has a base of convex sets

B. <u>Theorem</u>. Let $\mathcal{C}$ be a penetrating local convexity on a compact space X, with hereditary Hahn-Banach property.
(1). Then X has the usual fixed point property
(2) If X is normal then it has the Kakutani property

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locally compact metrizable

C. THEOREM Let $S_1, \ldots, S_n$ be strategy spaces of players $1, \ldots, n$, respectively. Let $\mathcal{C}_1$ be a convexity on S_1 and let $\succ_1 \subset S \times S$ (where $S = S_1 \times \cdots \times S_n$) be a pref. relation of player 1.

If $\mathcal{C}_1$ is compact-normal and has the compact-hull and
S_1 has a base of countably many convex sets
property while, $\succ$ is open (it suffices $\succ_1 \cap \Delta_1$ is rel. open in Δ_1, for $\Delta_1 := \{(s, s') \in S \times S \mid s_2 = s_2', \ldots, s_n = s_n'\}$) and, for all $s = (s_1, \ldots, s_n)$, $\{\xi \in S_1 \mid (\xi, s_2, \ldots, s_n) \succ s\}$ is convex-like then there exists a continuous function $P_1 : S_1 \times S \to \mathbb{R}$, ~~such~~ quasi-concave in S_1, s.t. for all $s_2, \ldots, s_n$ $P_1(\cdot, \cdot, s_2, \ldots, s_n)$ represents $\succ_1$.

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THEOREM. A game $(S_1, \ldots, S_n; P_1, \ldots, P_n)$ in which all S_i are compact metrizable spaces has an equilibrium whenever, for $i = 1, \ldots, n$, S_i admits a ~~local~~ normal penetrating convexity $\mathcal{C}_i$ with hereditary Hahn-Banach property, having the compact-hull property s. t. for all i, $s \in S$, $\{\xi \in S_i \mid (s_1, \ldots, \xi, \ldots, s_n) \succsim_i s\}$ is convex-like (in the sense of $\mathcal{C}_i$) and all P_i have open graphs.

COROLLARY 1. All this is good whenever all S_i are are compact metrizable sets in locally convex TVS

COROLLARY 2. All this is good whenever all S_i are metric spaces with connected sets taken as "convex"

The same holds if we consider spaces metrizable in a strongly convex way (in sense of Borsuk).

Define: equilibrium for a game.

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1. Convexity on a top. space X is any family of closed sets, closed under arbitrary intersections.

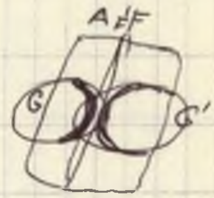
2. Convexity $\mathcal{K}$ is normal if for every $K \in \mathcal{K}$ and open set $F \supseteq K$, there exists $K' \in \mathcal{K}$ s.t. $F \supseteq K' \supseteq \mathrm{Int}\,K' \supseteq K$

3. Convexity $\mathcal{K}$ is local (or space is locally convex) if space has a base of convex sets.

4. If singletons are convex then normality implies local convexity.

5. Let $\mathcal{F}$ & $\mathcal{G}$ be families of subsets of a set X.

- $\mathcal{F}$ screens $\mathcal{G}$ if for every $A \in \mathcal{F}$, $G, G' \in \mathcal{G}$ s.t. $A \cap G \cap G' \neq \emptyset$ there exist $F, F' \in \mathcal{F}$ s.t. $A \cap G \subseteq F \setminus F'$ while $A \cap G' \subseteq F' \setminus F$



- $\mathcal{G}$ penetrates $\mathcal{F}$ if for any $F, F' \in \mathcal{F}$ with $F \cup F' \in \mathcal{F}$ and every $G \in \mathcal{G}$, if $G \cap F \neq \emptyset$ and $G \cap F' \neq \emptyset$ then $G \cap F \cap F' \neq \emptyset$.

6. Let $\mathcal{F}$ be a family of closed sets in a top. space. If inters. of every elt of $\mathcal{F}$ with every elt of $\mathcal{G}$ is connected then then $\mathcal{G}$ penetrates

*) Important special case: $\mathcal{F} = \mathcal{G} = \mathcal{K}$

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7. THEOREM. Let $\mathcal{K}$ be a convexity on a $\neq \emptyset$ compact space X. Assume $\exists$ a family $\mathcal{G}$ of subsets of X s.t. 1. $\mathcal{G}$ is finitely multiplicative, 2. $\mathcal{K}$ screens $\mathcal{G}$, 3. $\mathcal{G}$ penetrates $\mathcal{K}$ & 4. $\mathcal{G}$ is a topological base for X.
a. If X is locally convex then it has the f. p. property
b. If $\mathcal{K}$ is normal then the space ~~the~~ has the Kakutani property.

8. LEMMA. Let $\mathcal{F}$ and $\mathcal{G}$ be finitely multiplicative families of sets s.t. $\mathcal{F}$ screens $\mathcal{G}$ while $\mathcal{G}$ penetrates $\mathcal{F}$. If a finite indexed family α of elements of $\mathcal{G}$ covers a nonempty set $F \in \mathcal{F}$ then the nerve of the family $\alpha | F$ is a recursively contractible complex.

9. How about proofs? Divided into three parts!

10. Special cases:
a. Ky Fan - Glicksberg: X is a convex set in a TVS E
$\mathcal{K}$ - all closed convex sets.
We take $\mathcal{K} = \mathcal{F} = \mathcal{G}$, everything is fine if X - compact while E is locally convex
b. X is locally connected tree-like space, i.e. connected Hausdorff with "separation" property; then all closed connected sets are a convexity; if X is a tree (i.e. it is compact) then, by known results, $\mathcal{K}$ screens and penetrates itself. and therefore a tree has the Kak. property (Wallace Theorem).
c. A connecting f-n in a topol. space X is a cont. function $c : [0;1] \times X^2 \to X$ with $c(0, x, x') = c(1, x', x) =$
$= c(\alpha, x', x') = x'$.

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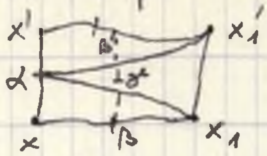
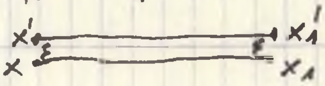
Convex sets defined as expected: $\mathcal{K}$ – this family penetrates itself

Condition →  then $\mathcal{K}$ screens itself

If space is metric and  then the convexity is normal.

Every compact metric space admitting a connecting function is an AR thus this theorem is a special
(Dugundji + Himmelberg, 1965) case of Eilenberg - Montgomery.

11. Products: Given family of topological spaces $(X_i \mid i \in I)$ with respective convexities $\mathcal{K}_i$.

Box convexity in $\Pi(X_i \mid i \in I)$ consists of all boxes