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Numerical methods and game-theoretic analysis of large scale individual production and consumption processes

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Abstract

The paper deals with a model of large scale individual production and consumption (households). The game-theoretic model involves infinitely many anonymous agents, classified into finitely many types depending on their efficiency and consumption patterns, producing, trading and consuming some goods. The model can be regarded as describing the behavior of an economy in one or two stages. Various solution concepts, such as competitive equilibrium, efficient production profiles, core and quasi-core are defined and studied. Finally, numerical experience with diversified special cases is reported.

Keywords and phrases: individual production, infinitely many agents, large model, household economy, equilibrium, core, efficiency

Numeryczne metody i teorio-growa analiza "dużych" procesów indywidualnej produkcji i konsumpcji

Streszczenie

W artykule zaprezentowany został model indywidualnej produkcji i konsumpcji uwzględniający działalność nieskończonej liczby anonimowych podmiotów, sklasyfikowanych w skończoną liczbę typów. Teorio-growy model opisuje zachowanie gospodarki w jednym lub dwóch okresach i bada pojęcia równowagi, rdzenia i efektywności. przedstawiono też numeryczną analizę wybranych przypadków szczególnych.

Słowa kluczowe: indywidualna produkcja, nieskończenie wielu producentów, "duży" model, gospodarka drobno-towarowa, równowaga, rdzeń, efektywność

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Individual or household production is an important subject in economic theory of labor markets, usually studied in the context of time allocation and jointly with questions of consumption. Many papers deal with the question of individual choice between non-market (home) and market production. Partial models may include some and miss some other elements mentioned above. The purpose of the present paper is to present a model of individual production and consumption (or household economy) which can be seen as a model of global but rather primitive economy yielding large scale output only after the process of aggregation, or as a partial economic model only dealing with questions of individual production. On the other hand, the model presented here can be used as an element in the construction of more complex models with elements not considered here. For simplicity but also in order to properly face some real world phenomena we do not fix or restrict the number of households but allow for the existence of infinitely many of them. Theoretical context of the questions considered in the present paper are included in Graham and Green 1984,Gronau 1980, 1986, Kerkhofs 1991 and other papers. The present paper offers a systematic survey and development of the results included in the papers of Wieczorek 1996, Roman (Ekes) and Wieczorek 1999, Ekes 1999 and Maćkiewicz and Wieczorek 2000. The paper is intended as a sort of survey of the results of the author and his collaborators and therefore the details of the proofs are not offered.

The model considered in the present paper deals with an infinite population of households classified into finitely many types, the types differ in their production skills and consumption patterns determined by demand functions. At the market there is a fixed number k of commodities each of which corresponds to an activity available to individuals of each type. However, individuals of different types usually differ in their productivity in different activities – typically most of their productivity coefficients applying to certain activities are zero (although no formal assumption of this kind is explicitly made). Given a price system, for each individual (household) of each type his income can be found for any kind of activity chosen by him. If prices were fixed properly, no shortage in the economy should occur which means that the total consumption determined by individual demand functions should does not exceed the total supply of all goods. Even a special case of the model with just one type of agents characterized by efficiency vector $(1, 1, ..., 1)$ exhibits some interest and it can be interpreted as a simple problem of differentiation of social activities in a uniform society. The more realistic case of many types of agents and more complex structure of coefficients of efficiency corresponds to the situation where certain individuals are more capable to fulfil one job than another and some individuals may simply occur more efficient than the others. The total output of this "household" production is somehow distributed among the members of the society. Our interest is to determine how can a socially stable distribution of activities be implemented by means of a market mechanism.

The model presented in this paper deals with a primitive economy in which the only source of individual income is the sale of individually produced commodities. The capital and initial endowment are not explicitly included in the model but, anyway, these factors, formally absent, may influence magnitude of the coefficients of individual efficiency of respective types of household. Larger coefficients of efficiency may correspond, for instance, to higher capital endowment or a possibility to engage physical endowments unavailable for other types of household. Differentiation of efficiency may also result from differentiation of individual skills or education.

In section 2 we give necessary conditions for the existence of a competitive equilibrium which consists of such a price system along with distributions of individual households of all types such that no individual could increase his income by changing his production activity. In section 3 we deal questions of efficiency applying to the allocation of individual effort: such an allocation is regarded as efficient whenever no other allocation of effort leads to a Pareto superior total output vector. Section 4 deals with concepts of the core and quasi-core of an economy, referring to actual, not necessarily balanced aggregated consumption of all households and, as it is usually the case with concepts like those, prices are found to exist to support certain states of the economy. Finally, in section 5, we quote the results of numerical calculations leading to competitive equilibria of models with specific data.

1 The Model of Individual Production and Consumption (Households)

The model of individual production and consumption (households) contains the following elements:

a positive integer n which is the number of *types of households*;

a positive integer k which is the number of different *activities* (equal to the number of *goods* at the market);

a vector $q = (q_1, q_2, \dots, q_n) \in \mathbb{R}^n_+$ describing the *structure of the population* of respective types, for convenience it may be assumed that $q \in \Delta_n$, the $(n-1)$-dimensional standard simplex in $\mathbb{R}^n$;

a matrix $R = \left(r^i_j\right)$ $(i = 1, \dots, n,\ j = 1, \dots, k)$, with nonnegative entries (matrix of *coefficients of efficiency*); it is understood that the household of type i undertaking the j-th activity produces r^i_j units of the good number j, for $i = 1, \dots, n$, $j = 1, \dots, k$;

(*demand*) functions $d^i : \mathbb{R}_+ \times \Delta_k \to \mathbb{R}^k_+$ of respective types of households; the actual demand of a household depends on its *income* $I = r^i_j \cdot \pi_j$, where π_j is the unit price of the j-th good and on the prevailing price system π; we assume that the equation $\langle d^i(I, \pi); \pi \rangle = I$ is satisfied for all arguments involved, i. e. the value of individual demand at given income at prevailing prices is equal to the actual income.

The so described economy will be shortly denoted by $\mathfrak{E}$.

A distribution of actions of the households of a fixed type, determined by a vector $p^i = \left(p^i_1, \dots, p^i_k\right) \in \Delta_k$, describes a situation where the fraction p^i_j (for instance understood as 25% if $p^i_j = 0.25$) of all households of type i decided to undertake the j-th activity, for $j = 1, \dots, k$ and $i = 1, \dots, n$. The sequence of vectors $\left(p^1, \dots, p^n\right)$, describing activity of households of all types will be shortly denoted by p.

Given a distribution of activities $\mathrm{p} = \left(p^1, \dots, p^n\right)$, the *aggregated demand* (for all goods) is the vector:

$$D(\mathrm{p}, \pi) = \sum_{i=1}^{n} q_i \sum_{j=1}^{k} d^i(r^i_j \cdot \pi_j, \pi) \cdot p^i_j,$$

while the *aggregated supply* (of all goods) is given by:

$$S(\mathrm{p}) = \left(\sum_{i=1}^{n} q_i \cdot r^i_1 \cdot p^i_1, \dots, \sum_{i=1}^{n} q_i \cdot r^i_k \cdot p^i_k \right).$$

2 Existence of competitive equilibria

A state of the economy $\mathfrak{E}$ is formally defined as a pair (p, π): a vector of distributions of activities of the households of all types $\mathrm{p} = \left(p^1, \dots, p^n\right)$ and a price vector π. A *competitive equilibrium* is a state of the economy (p, π) such that

$$D(\mathrm{p}, \pi) \leq S(\mathrm{p})$$

and for all $i = 1, \dots, n$ and all $j \in \operatorname{supp} p^i$ there is

$$r^i_j \cdot \pi_j = \max_{l=1,\dots,k} r^i_l \cdot \pi_l.$$

So a competitive equilibrium is a state of the economy, such that there is no good the demand for which is larger than its supply and such that (almost) all households are choosing activities giving them the maximal possible income, the same for all households of their type. We allow for an excess supply but this may only happen if the price of such a good is zero; this situation cannot happen at equilibrium if the demand functions (for each fixed good) are strictly decreasing in price of this good.

One way to prove the existence of a competitive equilibrium for the economy, used by Wieczorek in 1996 is to construct an auxiliary large game, with no large players and $n+1$ types of small players, each of whom has k available actions with appropriately defined payoff functions; players of the auxiliary $(n+1)$-st type are responsible for clearing the market, as it is often done in general equilibrium literature.

We have the following theorem (Wieczorek 1996):

Theorem 1 *Let $\mathfrak{E}$ be an economy with n types of agents.*
a. *Equilibrated distributions for the auxiliary game are the same as competitive equilibria for $\mathfrak{E}$.*
b. *If all demand functions d^i are continuous then the auxiliary game has an equilibrated distribution; hence there exists a competitive equilibrium for $\mathfrak{E}$.*
c. *(Walras Law) If all demand functions d^i are continuous and satisfy the condition*

$$d^i(I,\pi)\cdot\pi = I \text{ for all } I \text{ and } \pi \text{ in the domain of } d^i \tag{1}$$

then, at any competitive equilibrium (p^*,π^*), for all $j=1,\ldots,k$, $D_j(\mathrm{p}^*,\pi^*) < S_j(\mathrm{p}^*)$ implies $\pi_j^* = 0$.
d. *If all demand functions d^i are continuous and satisfy the condition (1) while all coefficients of efficiency are positive then, at any competitive equilibrium (p^*,π^*), there is $D(\mathrm{p}^*,\pi^*) = S(\mathrm{p}^*)$.*

If $\pi_j^* = 0$, for some j, then $S_j(\mathrm{p}^*) = 0$.

3 Efficiency and Weak Efficiency of Production Profiles

In this section we are interested in distributions p^* leading to supply vectors which are *efficient* [or *weakly efficient*] *in the sense of Pareto*, i.e. p^* such that there exists no other distribution p such that $S(\mathrm{p}) > S(\mathrm{p}^*)$ [respectively, $S(\mathrm{p}) \gg S(\mathrm{p}^*)$]. Usually such efficiency concepts are regarded as measuring efficiency of the organization of a society.

It is true that p is efficient if and only if there exists a system of positive prices $\pi = (\pi_1,\pi_2,\ldots,\pi_k)$ at which p^i maximizes the *total profit* (of all individuals) of type i, for each i (we speak of the total profit, however, this is achieved as a result of decentralized action of the players acting independently and having only their own profit in mind). More precisely, we have (Roman (Ekes) and Wieczorek 1999):

Theorem 2 *A distribution vector $\mathrm{p} = (p^1,p^2,\ldots,p^n)$ is Pareto efficient if and only if there exists a system of positive prices $\pi = (\pi_1,\pi_2,\ldots,\pi_k) \in \Delta_k$ at which p^i maximizes the total profit of type i, $d_i\sum_{j=1}^k r_j^i\pi_j p_j^i$, for each i.* ■

An analogue result concerning weak efficiency is the following:

Theorem 3 *A distribution vector $\mathrm{p} = (p^1,p^2,\ldots,p^n)$ is weakly Pareto efficient if and only if there exists a system of prices $(\pi_1,\pi_2,\ldots,\pi_k) \in \Delta_k$ at which p^i maximizes the total profit of the type i, $d_i\sum_{j=1}^k r_j^i\pi_j p_j^i$, for each i, and such that, for $j=1,\ldots,k$, $\pi_j > 0$ if and only if there exists no distribution q such that $S(\mathrm{q}) > S(\mathrm{p})$ and $(S(\mathrm{q}))_j > (S(\mathrm{p}))_j$.* ■

We also have the following:

Proposition 4 *If a distribution vector $\mathrm{p} = (p^1,p^2,\ldots,p^n)$ is weakly Pareto efficient and $(\pi_1,\pi_2,\ldots,\pi_k) \in \Delta_k$ is any system of prices at which p^i maximizes the total profit of the type i for each i then, for $j=1,\ldots,k$, $\pi_j = 0$ whenever there exists a distribution q such that $S(\mathrm{q}) > S(\mathrm{p})$ and $(S(\mathrm{q}))_j > (S(\mathrm{p}))_j$.* ■

The results presented in this section actually describe the process of decentralizing economic behavior of a society: efficient (or weakly efficient) states of an economy are rather obtained in a cooperative manner, an efficient state is jointly elaborated by all agents; in contrast, states at which individuals are maximizing their income have, *a fortiori*, noncooperative decentralized character. Such "decentralizing" results are known in many economic models for a long time (see e.g. Hildenbrand 1974, p. 232). Mathematical tools needed to prove the results in this section are mainly multicriterial optimization techniques applied to the set of all possible vectors of aggregated supply.

4 Core and Quasi-Core

The concept of competitive equilibrium depicts noncooperative stability of an economy, in turn the core always refers to possible cooperative behaviour of the agents; in particular, the core has no explicit reference to any price system. In this section we shall rather see the household model as a two-stage model: at the first stage the agents are producing goods while in the second they are consuming. The concept of the core is always related to the process of improvement of situations of the agents. Therefore the agents' preferences must be introduced somehow. In our model they may be revealed from demand functions or given explicitly; we take the latter approach.

To define formally the core of an economy and related concepts we need to modify, or rather enrich the original definition of the economy. In particular we introduce utility functions of individuals of respective types which are assumed to be coherent with initial demand functions, i. e. the demand functions arise by solving the problem of maximizing the utility subject to the constraint of staying in the budget set. Formally, a utility function $u : \mathbb{R}^k_+ \to \mathbb{R}$ is *coherent* with demand function $d : \mathbb{R}_+ \times \Delta_k \to \mathbb{R}^k_+$, whenever $u(d(I,\pi)) \geq u(\mathrm{x})$ for all $\mathrm{x} \in B(I,\pi) = \{\mathrm{x} \in \mathbb{R}^k_+ \mid \langle \mathrm{x}; \pi \rangle \leq I\}$. So in this case an *economy* $\mathfrak{E}$ is defined by natural numbers n, k, a vector $q \in \mathbb{R}^n_+$, matrix R and utility functions u^i.

To simplify the notation we write $\{1, \ldots, k\} = V$ and $W = V \times \mathbb{R}^k_+$; so W is the set of all pairs: individual action and individual consumption (in the case of disequilibrium, consumption may differ from the one determined by one's utility or demand function). A (*consumption inclusive*) *state* of the economy $\mathfrak{E}$ (*CI-state*, for short) is defined as a sequence $\mathrm{s} = (s^1, \ldots, s^n)$, whose elements are normed measures on $\mathfrak{B}(W)$ (Borel subsets of W).

A CI-state is said to be *admissible* whenever

$$\sum_{i=1}^n q_i \cdot \int_{\mathbb{R}^k_+} \mathrm{x} s^i (V \times d\mathrm{x}) \leq \left(\sum_{i=1}^n q_i \cdot r^i_1 \cdot s^i_1, \ldots, \sum_{i=1}^n q_i \cdot r^i_k \cdot s^i_k\right),$$

where $s^i_j = s^i(\{j\} \times \mathbb{R}^k_+)$ for $i = 1, \ldots, n$, $j = 1, \ldots, k$, i. e. when the aggregate consumption does not exceed the total production. (The integral in the above formula is taken with respect to the second marginal measure.)

Given a CI-state s, a *coalition* is understood as a sequence $\sigma = (\sigma^1, \ldots, \sigma^n)$ of measures on $\mathfrak{B}(W)$ such that $\sigma^i \leq s^i$ (i. e. $\sigma^i(A) \leq s^i(A)$ for all Borel sets A in W) holds for $i = 1, \ldots, n$. A coalition is *nonzero* if $\sigma^i(W) > 0$ for at least one i.

An *action* available for coalition $\sigma = (\sigma^1, \ldots, \sigma^n)$ is defined as a sequence $\mathrm{f} = (f^1, \ldots, f^n)$ of transition functions $f^i : W \times \mathfrak{B}(W) \to [0,1]$ (i. e. yielding a normed measure for every fixed value of the first variable and a measurable function for every fixed value of the second variable), for $i = 1, \ldots, n$. The composition of a measure σ on $\mathfrak{B}(W)$ with a transition function $f : W \times \mathfrak{B}(W) \to [0,1]$ will be denoted by $m = \sigma \circ f$.

An action $\mathrm{f} = (f^1, \ldots, f^n)$ is *admissible* for the coalition $\sigma = (\sigma^1, \ldots, \sigma^n)$ whenever

$$\sum_{i=1}^n q_i \cdot \int_{\mathbb{R}^k_+} \mathrm{x} m^i \left(V \times \mathbb{R}^k_+ \times V \times d\mathrm{x}\right) \leq$$

$$\left(\sum_{i=1}^n q_i \cdot r^i_1 \cdot m^i\left(V \times \mathbb{R}^k_+ \times \{1\} \times \mathbb{R}^k_+\right), \ldots, \sum_{i=1}^n q_i \cdot r^i_k \cdot m^i\left(V \times \mathbb{R}^k_+ \times \{k\} \times \mathbb{R}^k_+\right)\right).$$

An action f jest *favourable* for the coalition σ whenever, for $i = 1, \ldots, n$, the following holds:

$$m^i\left(\{(j, \mathrm{x}, j', \mathrm{x}') \in W \times W | u^i(\mathrm{x}') > u^i(\mathrm{x})\}\right) = m^i(W \times W),$$

where m^i denotes $\sigma^i \circ f^i$.

The *core* of the economy $\mathfrak{E}$ is defined as the set of its all admissible CI-states s such that there exist no nonzero coalition σ nor an admissible action f favourable for the coalition σ.

Quasi-core of the economy $\mathfrak{E}$ is defined as the set of its all admissible CI-states s such that there exist no nonzero coalition σ nor an admissible action f favourable for σ satisfying, for $i = 1, \ldots, n$, the condition:

$$\textstyle\sum_{j=1}^{k} m^i\left(\{j\} \times \mathbb{R}^k_+ \times \{j\} \times \mathbb{R}^k_+\right) = m^i(W \times W).$$

Description of an economy also including the consumption calls for a slight modification of the concept of equilibrium. To distinguish it from the former competitive equilibrium we shall call it simply equilibrium. An *equilibrium* is defined as a pair (s, π), where s is an admissible CI-state of the economy $\mathfrak{E}$, while π is a price system satisfying, for each $i = 1, \ldots, n$, the following condition:

$$s^i\left(\left\{(j, \mathrm{x}) \in W \middle| r^i_j \pi_j = \max_{l \in V} r^i_l \pi_l, \ \mathrm{x} = d^i\left(r^i_j \pi_j, \pi\right)\right\}\right) = 1.$$

Note that, if (s, π) is an equilibrium then, with notation $p^i_j = s^i\left(\{j\} \times \mathbb{R}^k_+\right)$, for $i = 1, \ldots, n$, $j = 1, \ldots, k$, the pair (p, π) is a competitive equilibrium at the distribution $\mathrm{p} = \left(p^1, \ldots, p^n\right)$.

The defined above concept of a CI-state includes an anonymous description of activities of all households and of the allocation of goods. While studying solutions which are in the core or quasi-core we do not take into consideration market mechanisms forming prices but only direct exchange of the goods within coalitions. Therefore the CI-state does not carry any information about prices in contrast to the usual state. On the other hand, we are interested in actual consumption of the households described, in the case of CI-states, by a distribution on the space of commodity bundles.

For a given CI-state, a coalition can be informally understood as a subset (because of anonymity not uniquely determined) of the set of all households of all types, making their decisions and consuming according to the description included in this state. Such a subset is described by means of measures σ^i satisfying the condition $\sigma^i \leq s^i$ for $i = 1, \ldots, n$. An action is a statistical description of changes of individual decisions and of reallocation of the goods within the coalition. An action is admissible for a coalition whenever aggregated consumption by this coalition is possible by the coalition's own production; an action is favourable whenever it gives outcomes which are better in the sense of utility function for all members of the coalition. So the elements of the core are just CI-states which no coalition can improve by changing both its production activities and actual consumption; in turn, in the case of quasi-core, such changes may only apply to reallocation of the goods and do not allow for changing initial production decisions. It follows that every element of the core must belong to quasi-core, but the inverse inclusion does not hold in general.

The theorem below states that, for a CI-state s in the core there exist prices which, along with distribution of the household actions derived from s form a competitive equilibrium in the model; in turn, if a pair (s, π) forms an equilibrium then s belongs to the quasi-core.

A utility function $u : \mathbb{R}^k_+ \to \mathbb{R}$ is said to be *essentially increasing* whenever, for all $x, y \in \mathbb{R}^k_+$ such that $x \ll y$ there is $u(x) < u(y)$.

We have the following theorem:

Theorem 5 *a. If a CI-state* s *belongs to the core while the utility functions u^i are essentially increasing for $i = 1, \ldots, n$, then there exist prices $\pi \in \Delta_k$ such that the state (p, π) is a competitive equilibrium, where $\mathrm{p} = (p^1, \ldots, p^n)$ is defined by $p^i_j = s^i\left(\{j\} \times \mathbb{R}^k_+\right)$, for all i, j.*

b. If a pair (s, π) is at equilibrium then the CI-state s *belongs to quasi-core.* ■

The theorem is proved by means of the following lemma (see Ekes 1999 for details).

Lemma 6 *If a CI-state* s *belongs to the core then the aggregate supply at this state, equal to*

$$\left(\sum_{i=1}^n q_i \cdot r_1^i \cdot s^i \left(\{1\} \times \mathbb{R}_+^k\right), \dots, \sum_{i=1}^n q_i \cdot r_k^i \cdot s^i \left(\{k\} \times \mathbb{R}_+^k\right)\right),$$

is weakly Pareto optimal. ■

Unlike in many other instances in general equilibrium theory, our model does not exhibit the full equivalence between the core and competitive equilibria, therefore we are forced to deal with two related core-like concepts: the core and the quasi-core. Definitions of a coalition and its actions are somewhat complicated from the formal point of view but they seem to preserve very well the anonymity property of the agents and, on the other hand, they seem to be quite intuitive.

5 Numerical Experience

The paper of Maćkiewicz and Wieczorek (2000) was dealing with special cases of household economies with specific numerical data; we quote them below. A program included in the Maćkiewicz's package LARGE_GAME_SOLVER (2000) was used to find competitive equilibria. In the sequel we shall use the following notation: for an n-vector $v = (v_1(\pi_1, \dots, \pi_n), \dots, v_n(\pi_1, \dots, \pi_n))$ depending on real parameteres $\pi_1, \dots, \pi_n$ we write $\Downarrow v \Downarrow$ to denote the n-vector $v \cdot \langle v; (\pi_1, \dots, \pi_n)\rangle^{-1}$ if the inner product $\langle v; (\pi_1, \dots, \pi_n)\rangle$ is nonzero and $\Downarrow v \Downarrow = v$ otherwise. The specific data for calculations were fixed as indicated below:

Cases 1 and 2 include two types of households and three activities/commodities, in both cases the structure of the population is $(0.4, 0.6)$; the coefficients of efficiency are the same in both cases: $(r_1^1, r_2^1, r_3^1) = (2, 6, 5)$ and $(r_1^2, r_2^2, r_3^2) = (4, 3, 4)$. The two cases only differ in demand functions:

In case 1 the demand function for type i, $i = 1, 2$, has the form:

$$d^i(I, \pi) = I \cdot \Downarrow\left(b_1^i 3 \left(\pi_1 + \varepsilon_1^i\right)^{-a_1^i}, b_2^i 2 \left(\pi_2 + \varepsilon_2^i\right)^{-a_2^i}, b_3^i 4 \left(\pi_3 + \varepsilon_3^i\right)^{-a_3^i}\right)\Downarrow.$$

To perform calculations, we took a specific set of data:
$b_1^1 = 3, b_2^1 = 2, b_3^1 = 4, a_1^1 = a_2^1 = a_3^1 = 1, \varepsilon_1^1 = \varepsilon_1^1 = \varepsilon_1^1 = 0.001$ and
$b_1^2 = 4, b_2^2 = 3, b_3^2 = 4, a_1^2 = a_2^2 = a_3^2 = 1, \varepsilon_1^2 = \varepsilon_1^2 = \varepsilon_1^2 = 0.001$.

In case 2 the demand function for type i, $i = 1, 2$, has the form:

$$d^i(I, \pi) = I \cdot \Downarrow\left(b_1^i 3 \left(\max(\pi_1, \varepsilon_1^i)\right)^{-a_1^i}, b_2^i 2 \left(\max(\pi_2, \varepsilon_2^i)\right)^{-a_2^i}, b_3^i 4 \left(\max(\pi_3, \varepsilon_3^i)\right)^{-a_3^i}\right)\Downarrow;$$

To perform calculations, we took a specific set of data:
$b_1^1 = 3, b_2^1 = 2, b_3^1 = 4, a_1^1 = a_2^1 = a_3^1 = 1, \varepsilon_1^1 = \varepsilon_1^1 = \varepsilon_1^1 = 0.001$ and
$b_1^2 = 4, b_2^2 = 3, b_3^2 = 4, a_1^2 = a_2^2 = a_3^2 = 1, \varepsilon_1^2 = \varepsilon_1^2 = \varepsilon_1^2 = 0.001$
(note that the parameters are exactly the same as in case 2; the only difference is the formula defining the demand function).

The numerically found competitive equilibria in case 1 were always the same, for several different starting points, namely we got:
$(p_1^1, p_2^1, p_3^1) = (0.0000000000172729, 0.5885702446190412, 0.4114297553636859)$;
$(p_1^2, p_2^2, p_3^2) = (0.4934199793304743, 0.0000000000458280, 0.5065800206236977)$;
$(\pi_1, \pi_2, \pi_3) = (0.3529411764723749, 0.2941176470584993, 0.3529411764691258)$.

Also the numerical results obtained in case 2 were always the same, for several different starting points, namely:

$(p_1^1, p_2^1, p_3^1) = (0.0000000012192975, 0.5887700530395075, 0.4112299457411950);$
$(p_1^2, p_2^2, p_3^2) = (0.4933155037203975, 0.0000000000926676, 0.5066844961869349);$
$(\pi_1, \pi_2, \pi_3) = (0.3529411780961891, 0.2941176471833995, 0.3529411747204114).$

The obtained results are nearly the same which can be explained by the fact that in the neighborhood of the found equilibrium (which is probably unique in both cases, 1 and 2) the demand functions in cases 1 and 2 are nearly identical even though the defining formulae are different.

Cases 3 and 4 include three types of households and two activities/commodities, in both cases the structure of the population is $(0.3, 0.6, 0.1)$; the coefficients of efficiency are the same in both cases: $(r_1^1, r_2^1) = (2, 6)$ and $(r_1^2, r_2^2) = (4, 3)$. The two cases only differ in demand functions (the demand function in case 3 is elliptic, in case 4 it is truncated Cobb-Douglas):

In case 3 the demand function for type i, $i = 1, 2, 3$, has the form:

$$\begin{aligned} d^i(I,\pi) &= I \cdot \Big\Downarrow \Big(a_1^i + \left(e_1^i\right)^{-1} - \left(e_1^i\right)^{-2} \pi_1 \left(e_1^{-2} (\pi_1)^2 + e_2^{-2} (\pi_2)^2\right)^{-\frac{1}{2}}, \\ & \qquad a_2^i + \left(e_2^i\right)^{-1} - \left(e_2^i\right)^{-2} \pi_1 \left(e_2^{-2} (\pi_1)^2 + e_2^{-2} (\pi_2)^2\right)^{-\frac{1}{2}} \Big) \Big\Downarrow . \end{aligned}$$

To perform calculations, we took a specific set of data:
$a_1^1 = 0.41(6), a_2^1 = 0.8, e_1^1 = 1.2, e_2^1 = 1,$
$a_1^2 = 1, a_2^2 = 0.2, e_1^2 = 0.8, e_2^2 = 1$ and
$a_1^3 = 0.6, a_2^3 = 0.4, e_1^3 = 1, e_2^3 = 1.$

In case 4 the demand function for type i, $i = 1, 2, 3$, has the form:

$$d^i(I,\pi) = I \cdot \Downarrow \delta^i(\pi) \Downarrow,$$

where

$$\delta^i(\pi) = \left(\delta_1^i(\pi), \delta_2^i(\pi)\right) = \begin{cases} \left(\frac{1}{\pi_1+\pi_2 e_1^i}, \frac{e_1^i}{\pi_1+\pi_2 e_1}\right) & \text{if } \frac{\pi_1}{\pi_2} < \frac{a_1^i e_1^i}{a_2^i}, \\ \left(\frac{a_1}{\pi_1\left(a_1^i+a_2^i\right)}, \frac{a_2^i}{\pi_2(a_1+a_2)}\right) & \text{if } \frac{a_1^i e_1^i}{a_2^i} \leqslant \frac{\pi_1}{\pi_2} \leqslant \frac{a_1^i e_2^i}{a_2^i}, \\ \left(\frac{1}{\pi_1+\pi_2 e_2^i}, \frac{e_2^i}{\pi_1+\pi_2 e_2^i}\right) & \text{if } \frac{\pi_1}{\pi_2} > \frac{a_1^i e_2^i}{a_2^i} \text{ or } \pi_2 = 0. \end{cases}$$

To perform calculations, we took a specific set of data:
$a_1^1 = 1, a_2^1 = 1.2, e_1^1 = 2, e_2^1 = 7,$
$a_1^2 = 1.3, a_2^2 = 1.1, e_1^2 = 3, e_2^2 = 5$ and
$a_1^3 = 1, a_2^3 = 1.5, e_1^3 = 0.4, e_2^3 = 1.2.$

The numerical results obtained in case 3 were (probably unique, for several starting points):
$(p_1^1, p_2^1) = (0.0000000000000000, 1.000000000000000);$
$(p_1^2, p_2^2) = (1.000000000000000, 0.0000000000000000);$
$(p_1^3, p_2^3) = (0.4144438621787334, 0.5855561378212666);$
$(\pi_1, \pi_2) = (0.4444444444442955, 5555555555557046).$

The numerical results obtained in case 4 were (probably unique, for several starting points):
$(p_1^1, p_2^1) = (0.0000000000000000, 1.000000000000000);$
$(p_1^2, p_2^2) = (1.000000000000000, 0.0000000000000000);$
$(p_1^3, p_2^3) = (0.2559967789055940, 0.7440032210944061);$
$(\pi_1, \pi_2) = (0.4444444439975023, 0.5555555560024977).$

Note that the results obtained in cases 3 and 4 are nearly the same except (p_1^3, p_2^3) which are distributions of strategies of the third type of individuals. This outcome was not expected, although it does not contradict economic intuitions.

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