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Abstract · Streszczenie

The paper deals with a modification of the Shapley value in which the process of coalition formation is not necessarily uniform (as in the case of the usual value) but it depends on some indicators of individual initiative and attraction toward already existing coalitions. The value is given an axiomatic treatment and some methods are proposed how to measure the coefficients of initiative and attraction of individuals in a given population by comparing the expected and actual outcomes in a sequence of games played within this population. Presented example deals with the population of students participating at an experiment performed at Warsaw University.

METODA POMIARU INDYWIDUALNEJ INICJATYWY I ATRAKCJI PRZY POMOCY ZMODYFIKOWANEJ WARTOSCI SHAPLEYA

W pracy zdefiniowano modyfikację pojęcia wartości Shapleya, w przypadku, kiedy proces tworzenia się koalicji nie jest równomierny (jak w przypadku zwykłej wartości) lecz zależy od pewnych wskaźników indywidualnej inicjatywy i skłonności do przystępowania do już istniejących koalicji. Wartość jest tu potraktowana w sposób aksjomatyczny. Zaproponowano metodę pomiaru współczynników indywidualnej inicjatywy i atrakcji w danej populacji przez porównanie oczekiwanych i faktycznych wyników serii gier rozgrywanych w tej populacji. Podano przykład populacji studentów Uniwersytetu Warszawskiego biorących udział w eksperymencie.

Key phrases: Shapley value, measuring initiative and attraction

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In the present paper we study cooperative games with indicators of initiative and attraction which are defined as pairs: a usual characteristic function game v and and a system γ of numbers describing propensities of the individuals to join the already formed coalitions. The coefficient $\gamma_i^{\emptyset}$ is then interpreted as a measure of initiative of the individual i; for $S\neq\emptyset$ and $i\notin S$, γ_i^S is a measure of attraction of the individual i toward a coalition S.

For games of this kind, we axiomatically define the notion of value; we prove its existence and uniqueness giving an explicit formula. In case of the uniform indicators of initiative and attraction $\gamma_i^S=|N\setminus S|^{-1}$, the value coincides with the usual Shapley value.

The theory can be applied to determine the indicators of initiative and attraction for a given set of individuals. One can record outcomes in a series of games played by these players and then try to find indicators best fitting the observed outcomes. The phrase "best fitting" can be understood in many different ways and we propose and study a few of them.

We illustrate the procedure by example concerning the results of an experiment performed at Warsaw University (three students were asked to play a series of cooperative games).

Finally, we discuss relations of our results to those existing in the literature.

I. Games with indicators of initiative and attraction

Let $N = \{1,2,\cdots,n\}$ be a fixed set of players. We shall be dealing with usual characteristic function games over N, in this paper called *power functions* defined as functions v assigning to each coalition $S \subseteq N$ a real number $v(S)$ and such that $v(\emptyset)=0$. The set of all power functions over N will be denoted by V.

Denote $A := \{(S,i) \mid S \subset N,$ and $i \in S\}$ (here and elsewhere in the paper the sign "$\subset$" denotes the strict inclusion). A *system of indicators of initiative and attraction* on N, abbreviated to *system of indicators* is defined as a function $\gamma: A \longrightarrow \mathbb{R}_+$ assigning to each pair $(S,i) \in A$ a nonnegative number γ_i^S, also denoted by $\gamma(S,i)$, so that for all $S \neq N$, $\Sigma_{i \in S} \gamma_i^S \leq 1$. The set of all systems γ will be denoted by Γ.

A *cooperative game with indicators of initiative and attraction* is defined as a triple

$$G = \langle N, v, \gamma \rangle,$$

where N is the set of players, v is a power function, and γ is a system of indicators of initiative and attraction on N.

A *solution concept* for cooperative games G is a function $\Psi: V \times \Gamma \longrightarrow \mathbb{R}^n$. For any $(v,\gamma) \in (V \times \Gamma)$, we shall write

$$\Psi(v,\gamma) = [\Psi_1(v,\gamma), \Psi_2(v,\gamma), \cdots, \Psi_n(v,\gamma)].$$

We shall be interested in a solution concept satisfying some particular postulates. Before their formulation we must introduce some definitions.

Let $i \in N$ and $v \in V$. A coalition $S \subseteq N \setminus i$ is *i-essential for v* if $v(S \cup i) \neq v(S)$. Player i is *null* in the game v if no coalition $S \subseteq N \setminus i$ is i-essential for v.

Let $i \in N$ and $S \subseteq N \setminus i$. The Player i is said to be *S-null* for a power function v if $v(T \cup i) = v(T)$ for all $T \subseteq N \setminus i$ with $T \cap S \neq \emptyset$.

Let $S \subset N$. A system of indicators γ is said to be *S-exhausting* if $\Sigma_{i \in S \setminus T} \gamma_i^T = 1$ for all $T \subset S$ and if $\gamma_i^T = 0$ whenever $i \in S$ and $T \cap (N \setminus S) \neq \emptyset$.

We say that systems of indicators γ and $\bar{\gamma}$ are *consistent on* a set $S \subset N$ if $\gamma_k^R = \bar{\gamma}_k^R$ whenever $k \in S$, and $R \subseteq S \setminus k$.

The definition of a null player is standard. Further, a coalition S may wish to admit only i-essential players. On the other hand, if Player i is S-null for some S, only subcoalitions of $N\backslash S\backslash i$ may be interested in including that player to increase their power. At the end, notice that, if a system of indicators is S-exhausting, then only subcoalitions of the coalitions S or $N\backslash S$ can effectively arise.

Let $k\in N$. Let γ and $\bar{\gamma}$ be systems of indicators satisfying $\gamma_j^S=\bar{\gamma}_j^S$ for all S whenever $j\neq k$. We define a real function $\gamma \overset{k}{+} \bar{\gamma}$ on A in the following way:

$$[\gamma \overset{k}{+} \bar{\gamma}](T,i) = \begin{cases} \gamma_i^T & \text{if } i\neq k, \\ \gamma_k^T + \bar{\gamma}_k^T & \text{if } i=k. \end{cases}$$

(Note that under obvious inequality restrictions, $\gamma \overset{k}{+} \bar{\gamma}$ is also a system of indicators.)

We define the *value* Φ as a solution concept satisfying the following six postulates:

Axiom 1 (*v-linearity*): For each $\gamma\in\Gamma$, $v,w\in V$ and $\alpha,\beta\geq 0$,

$$\Phi(\alpha v+\beta w,\gamma) = \alpha\Phi(v,\gamma) + \beta\Phi(w,\gamma).$$

Axiom 2 (*null player*): If i is a null player for $v\in V$, then $\Phi_i(v,\gamma) = 0$ for all $\gamma\in\Gamma$.

Axiom 3 (*efficiency*): Let $S\subseteq N$ and $\gamma\in\Gamma$. If γ is S-exhausting, then for any power function v

$$\Sigma_{i\in S}\, \Phi_i(v,\gamma) = v(S).$$

Axiom 4 (*equivalence*): Let $S\subset N$, $i\in N\backslash S$ and $v\in V$. If γ and $\bar{\gamma}$ are systems consistent on S and satisfying $\gamma_i^S= \bar{\gamma}_i^S$, and if player i is $N\backslash S\backslash i$-null for v, then

$$\Phi_i(v,\gamma) = \Phi_i(v,\bar{\gamma}).$$

Axiom 5 (*boudedness*): For each power function v the function $\Phi(v,\cdot)$ is bounded on Γ.

Axiom 6 (γ-*additivity*): Let $i,k \in N$ and $v \in V$. Let γ and $\bar{\gamma}$ be such systems that $\gamma_j^S = \bar{\gamma}_j^S$ for all S whenever $j \neq k$, and $\gamma \overset{k}{+} \bar{\gamma} \in \Gamma$. If one of the conditions:

a. $k=i$; or

b. Player k belongs to all i-essential coalitions for v;

is satisfied then

$$\Phi_i(v,\gamma \overset{k}{+} \bar{\gamma}) = \Phi_i(v,\gamma) + \Phi_i(v,\bar{\gamma}).$$

Axioms 1 and 2 are standard. Axiom 5 is obvious. Also the interpretation of Axiom 3 is clear. Namely, when γ is S-exhausting, no coalition properly containing S is willing to arise. Therefore, it is reasonable to require that the whole worth $v(S)$ should be divided only between the players in S.

Assume now that a Player i is $N\backslash S\backslash i$-null. Then it may be useful for that Player only to join subcoalitions of S. Having this in mind, the requirement of Axiom 4 seems very natural.

The interpretation of Axiom 6 is based on some further assumptions about the model. Let Player $k \neq i$ belong to all i-essential coalitions. Hence, it may be useful for Player i only to join coalitions T containing Player k. On the other hand, during the process of creating such a coalition T, Player k influences that event by his earlier joining subcoalitions of T (with respective intensities γ_k^S, $S \subseteq T\backslash k$). Now, if we assume that Player k influences the payoff of Player i by joining T in an "additive way", and that the whole payoff of Player i is the sum of all his payoffs related to all such coalitions T, then Axiom 6 is a natural consequence of that fact. If $k=i$, the interpretation is similar.

For $\gamma \in \Gamma$ and $T \subseteq N$ we introduce the following notation:

$$q(\gamma,\emptyset) := 1, \tag{1}$$

and for $T \neq \emptyset$

$$q(\gamma,T) = \sum_{\bar{\iota} \in \Pi(T)} Q(\gamma,\bar{\iota}), \tag{2}$$

where $\Pi(T)$ denotes the set of all orderings $\bar{\iota}=(\iota_1,\cdots,\iota_t)$ of T,

$$Q(\gamma,\bar{\iota}) = \gamma(\emptyset,\iota_1)\gamma(\{\iota_1\},\iota_2)\times\cdots\times\gamma(\{\iota_1,\cdots,\iota_{t-1}\},\iota_t) \tag{2a}$$

while $t=|T|$.

Clearly, if γ_i^S is interpreted as the probability that Player i joins the previously existing coalition S then $q(\gamma,T)$ corresponds to the probability of formation of coalition S at all.

Theorem. *There exists a unique value* $\Phi(v,\gamma)=[\Phi_1(v,\gamma),\cdots,\Phi_n(v,\gamma)]$ *on the class of all cooperative games G with indicators of initiative and attraction which satisfies* Axioms 1-6, *given by*

$$\Phi_i(v,\gamma) = \sum_{T\subseteq N\setminus i} q(\gamma,T)\gamma_i^T[v(T\cup i) - v(T)], \qquad (3)$$
$$i\in N,\ v\in V,\ \gamma\in\Gamma.$$

Substitutitng in (3) the uniform system $\bar{\gamma}_i^S := |N\setminus S|^{-1}$, we immediately obtain:

Corollary. *For each power function* v, $\Phi(v,\bar{\gamma})$ *is the Shapley value of* v.

II. Proof of the theorem

The proof of the Theorem will be preceeded by several lemmata:

Lemma 1. *The value* Φ *determined by* Theorem 1 *satisfies* Axioms 1-6.

Proof. The fact that Φ satisfies Axiom 1,2 and 5 is trivial; in case of the remaining axioms it is a matter of easy verification.

Lemma 2. *Every value* $\Phi=[\Phi_1,\Phi_2,\cdots,\Phi_n]$ *satisfying* Axioms 1 *and* 2 *has of the form:*

$$\Phi_i(v,\gamma) = \sum_{T\subseteq N\setminus i} a_i^T(\gamma)[v(T\cup i) - v(T)],\quad i\in N,\ v\in V,\ \gamma\in\Gamma, \qquad (4)$$

where $a_i^T(\gamma)$ *are some constants independent of* v.

Proof. It proceeds in exactly the same way as for Theorems 1 and 2 in Weber [1988]. One should only replace the dummy axiom by Axiom 2, and the "dummy player" by "null player".

Lemma 3. *Let f be a bounded function from a rectangle B (in $\mathbb{R}^\ell$) into $\mathbb{R}$, which satisfies: $f(x+y) = f(x)+f(y)$ for all $x,y\in B$ with $x+y\in B$. Then there exist constants $a_1,a_2,\cdots,a_\ell$ such that for each $x=(x_1,x_2,\cdots,x_\ell)\in B$, $f(x) = a_1x_1+a_2x_2+\cdots+a_\ell x_\ell$.*

Proof. The case $\ell=1$ is classical (the known Cauchy functional equation). The generalization to $\ell>1$ is straightforward. Therefore $f(0,\cdots,x_i,\cdots,0) = a_ix_i$ for $i=1,\cdots,\ell$. Summing now over all i, the lemma follows.

Lemma 4. *Let a value $\Phi=[\Phi_1,\Phi_2,\cdots,\Phi_n]$ have the form* (4) *and satisfy part* (a) *of* Axiom 6 *and* Axiom 4. *Then for each $\gamma\in\Gamma$ there exist constants $(b_i^T(\gamma)\,|\ i\in N,\ T\subset N\setminus i)$ such that for all $T\neq N$*

$$a_i^T(\gamma) = b_i^T(\gamma)\gamma_i^T\ , \qquad for \quad i\in N\setminus T, \tag{5}$$

and for any two systems γ and $\bar{\gamma}$ consistent on T,

$$b_i^T(\gamma) = b_i^T(\bar{\gamma}), \qquad \text{for} \quad i\in N\setminus T. \tag{6}$$

Proof. Let us fix $i\in N$ and $R\subseteq N\setminus i$. Let u be the power function determined by: $u(T)=0$ for $T\neq R\cup i$ and $u(R\cup i)=1$. The only i-essential coalition for u is R. Hence, (4) implies

$$\Phi_i(u,\gamma) = a_i^R(\gamma), \qquad \text{for } \gamma\in\Gamma. \tag{7}$$

Now, let γ and $\bar{\gamma}$ be any two systems consistent on R and satisfying $\gamma_i^R=\bar{\gamma}_i^R$. Since Player i is $N\setminus R\setminus i$-null for u, Axiom 4 and (7) imply $a_i^R(\gamma)=a_i^R(\bar{\gamma})$. Hence, by the arbitrarity of γ and $\bar{\gamma}$, it follows that $a_i^R(\gamma)$ depends only on the variable γ_i^R and on the variables from the set $D^R(\gamma)$, where

$$D^R(\gamma) := \{\gamma_j^T \mid j\in R,\ T\subset R\setminus j\}\ . \tag{8}$$

On the other hand, we easily see that (a) of Axiom 6 can be applied to get that $\Phi_i(u,\gamma)$ is additive in γ_i^R. Therefore, because of (7), also $a_i^R(\gamma)$ is additive in γ_i^R, and Lemma 3 applied to this fact immediately implies (5). Relation (6) is a straightforward consequence of the statement given directly before (8). This completes the proof of the lemma.

To formulate the next lemma we need some notation:

Let $i \in N$. For $T \subseteq N \setminus i$, we define

$$\Lambda(T) := \{\varphi \mid \varphi : T \to 2^T \text{ satisfies } j \notin \varphi(j) \text{ for all } j \in T\}$$

(in particular, $\Lambda(\emptyset) = \emptyset$) and, for $i \in N$, let $\Lambda_i := \bigcup_{T \subseteq N \setminus i} \Lambda(T)$.

Further, for $\gamma \in \Gamma$, $i \in N$ and a mapping $E_i : \Lambda_i \to \mathbb{R}_+$, let

$$\rho_i(\gamma, \emptyset, E) := E(\emptyset, i) \tag{9}$$

and for $T \neq \emptyset$,

$$\rho_i(\gamma, T, E_i) := \sum_{\varphi \in \Lambda(T)} E_i(\varphi)\gamma(\varphi(x_1^T), x_1^T)\gamma(\varphi(x_2^T), x_2^T) \times \cdots \times \gamma(\varphi(x_t^T), x_t^T), \tag{10}$$

where $T = \{x_1^T, \cdots x_t^T\}$.

Lemma 5. *Assume that a value* $\Phi = [\Phi_1, \cdots, \Phi_n]$ *satisfies* Axioms 1, 2 *and* 4-6. *Then for each* $i \in N$ *there exists a function* $E_i : \Lambda_i \to \mathbb{R}_+$ *such that*

$$\Phi_i(v, \gamma) = \sum_{T \subseteq N \setminus i} \rho_i(\gamma, T, E_i)\gamma_i^T[v(T \cup i) - v(T)], \qquad i \in N,\ v \in V,\ \gamma \in \Gamma, \tag{11}$$

with ρ_i *defined by* (9) *and* (10).

Proof. Let us fix $i \in N$, $R \subseteq N \setminus i$, and $k \in R$. Let u be the power function defined at the beginning of the proof of Lemma 4. Therefore, by (7) and (5),

$$\Phi_i(u, \gamma) = b_i^R(\gamma)\gamma_i^R, \qquad \text{for } \gamma \in \Gamma, \tag{12}$$

and (6) holds for $T = R$. The coalition R is the only i-essential coalition for u. Hence, since $k \in R$, Player k belongs to all i-essential coalitions for u. Because of (6), $b_i^R(\gamma)$ depends only on $\gamma_j^T \in D^R(\gamma)$ (see (8)). Let us fix for a moment the variables γ_i^R and all from the set

$$D_{-k}^R(\gamma) := \{\gamma_j^T \mid j \in R,\ j \neq k,\ T \subseteq R \setminus j\} ,$$

and consider $\Phi_i(u,\gamma)$ as a function of variables from the set

$$D^R_k(\gamma) := D^R(\gamma) \setminus D^R_{-k}(\gamma).$$

Condition (b) of Axiom 6 says that this function is additive. Therefore Lemma 3 can be applied to $\Phi_i(u,\gamma)$ w.r.t. variables from $D^R_k(\gamma)$ to get the following formula with the help of (12):

$$b^R_i(\gamma) = \sum_{T\subseteq R\setminus k} c^T_{ik}(\gamma)\, \gamma^T_k, \qquad \text{for } \gamma\in\Gamma, \tag{13}$$

where coefficients $c^T_{ik}(\gamma)$ depend only on the variables from the set $D^R_{-k}(\gamma)$. Now we fix $S\subseteq R\setminus k$ and $l\in R\setminus k$. For further considerations we shall need the following two sets

$$\Gamma^S_k := \{\gamma\in\Gamma \mid \gamma^T_k = 0 \text{ for } T\neq S\},$$

and

$$D^R_{-k-l}(\gamma) := \{\gamma^T_j \mid j\in R,\ j\neq k,\ j\neq l,\ T\subseteq R\setminus j\} .$$

By (12) and (13), it is easily seen that the restriction of the function $\Phi_i(u,\cdot)$ to the set Γ^S_k satisfies

$$\Phi_i(u,\gamma) = c^S_{ik}(\gamma)\, \gamma^S_k\, \gamma^R_i\,, \qquad \gamma\in\Gamma^S_k. \tag{14}$$

Let us fix for a moment the variables from the set $D^R_{-k-l}(\gamma)$ and consider $\Phi_i(u,\gamma)$ as a function $\bar\Phi_i$ of variables from the set

$$D^R_{kl}(\gamma) := D^R_{-k}(\gamma) \setminus D^R_{-k-l}(\gamma).$$

Now we easily see that Axiom 6 with (b) holds under $\Phi_i \equiv \bar\Phi_i$ and $k\equiv l$ again. Therefore, it follows that $\Phi_i(u,\gamma)$ is additive in variables from $D^R_{kl}(\gamma)$, and consequently, Lemma 3 with the help of (12) and (13) give the following formula:

$$c^S_{ik}(\gamma) = \sum_{T\subseteq R\setminus l} d^{ST}_{ikl}\, \gamma^T_l \qquad \text{for } \gamma\in\Gamma, \tag{15}$$

where coefficients d^{ST}_{ikl} depend only on the variables from the set $D^R_{-k-l}(\gamma)$. Now, taking into account (13) and (15), we get

$$b^R_i(\gamma) = \sum_{T\subseteq R\setminus k}\ \sum_{U\subseteq R\setminus l} d^{TU}_{ikl}\,\gamma^U_l\,\gamma^T_k \qquad \text{for } \gamma\in\Gamma.$$

It is easily seen that this procedure can be continued to the moment when each of the coefficients obtained at the end will be constant on the set Γ. Then these coefficients will be equal to $E_i(\varphi)$ for $i\in N$, with the obvious $\varphi\in\Lambda(R)$ and, consequently, $b^R_i(\gamma) = \rho_i(\gamma,R,E_i)$, according to (10). This equality also holds in the second case $R=\emptyset$ because of (6) for $T=\emptyset$ and (9). Now formula (11) is a simple consequence of Lemmata 2 and 4. Thus the lemma has been proved.

Lemma 6. *Let $\emptyset\neq T\subseteq N$ and $j\in T$. Assume that for all T-exhausting $\gamma\in\Gamma$*

$$\sum_{\varphi\in\Lambda(T\setminus j)} c(\varphi)\gamma(\varphi(x^T_1),x^T_1)\gamma(\varphi(x^T_2),x^T_2)\times\cdots\times\gamma(\varphi(x^T_k),x^T_k) = 0, \tag{16}$$

where $c(\varphi)$ are real constants while $T\setminus j = \{x^T_1,\dots,x^T_k\}$. Then

$$c(\varphi) = 0 \quad \text{for} \quad \varphi\in\Lambda(T). \tag{17}$$

Proof. Note that any system $(\gamma^R_i \mid i\in T\setminus j,\ R\subset T)$ with all elements contained in a sufficiently small interval $(0,a)$, $a>0$, can be extended to some T-exhausting system $\gamma\in\Gamma$. Therefore, the right-hand side of (16) can be considered as polynomial which is equal to 0 on some small rectangle. Hence, its coefficients must satisfy (17).

Lemma 7. *Let $\Phi = [\Phi_1,\Phi_2,\dots,\Phi_n]$ be a value determined by* Lemma 5, *which satisfies* Axiom 3. *Then Φ coincides with the value determined by* (1)-(3).

Proof. Let Φ of the form (11) satisfy the assumption of the lemma. Let $i\in N$ and define E^0_i letting

$$E^0_i(\varphi) = \begin{cases} 1 & \text{if } \varphi \text{ is one-to-one and the family of sets} \\ & \{\varphi(m)\mid m\in\mathrm{Dom}\,\varphi\} \text{ is linearly ordered by inclusion;} \\ 0 & \text{otherwise}, \end{cases}$$

(notice that $E^0_i(\emptyset) = 1$ for all i).

One can easily see that the value $\Phi^{0}=[\Phi^{0}_{1},\cdots,\Phi^{0}_{n}]$ of the form

$$\Phi^{0}_{i}(v,\gamma) = \sum_{T\subseteq N\setminus i} \rho_{i}(\gamma,T,E^{0}_{i})\gamma^{T}_{i}[v(T\cup i) - v(T)], \qquad (18)$$

$$i\in N,\ v\in V,\ \gamma\in\Gamma,$$

with ρ_{i} defined by (9) and (10), coincides with the value Φ described by (1)-(3). Therefore it suffices to show that $\Phi=\Phi^{0}$, or equivalently that

$$E_{i}(\rho) = E^{0}_{i}(\rho), \quad \text{for } i\in N,\ T\subseteq N\setminus i,\ \rho\in\Lambda(T). \qquad (19)$$

We shall give an inductive proof of these equalities with respect to the number $t=|T|$.

Let $i\in N$. Further, let $\bar{\gamma}^{\emptyset}_{i} = 1$ and $\bar{\gamma}^{S}_{j}=0$ for $(S,j) \neq (\emptyset,i)$. Hence, (11) implies that $\Phi_{i}(v,\bar{\gamma}) = E_{i}(\emptyset)v(i)$ for $v\in V$. On the other hand, $\bar{\gamma}$ is $\{i\}$-exhausting. Therefore, by Axiom 3, $\Phi_{i}(v,\bar{\gamma}) = v(i)$ for $v\in V$. Thus $E_{i}(\emptyset)=1$ and (17) has been shown for $t=0$.

Assume now that (17) holds under the additional assumption $|T|\leq k-1$, where k is some integer $1\leq k<n$. (Note that this implies $\rho_{i}(\bar{\gamma},T,E^{0}) = \rho_{i}(\bar{\gamma},T,E)$ if $|T|\leq k-1$). We shall show that this implies the validity of (17) when $|T|=k$.

Let $R\subseteq N$ with $|R|=k+1$ be fixed. Let $\bar{\gamma}$ be any R-exhausting system such that $\bar{\gamma}^{R}_{i} = 0$ for $i\in R$. Hence, by (10),

$$\rho_{i}(\gamma,T,E_{i}) = \rho_{i}(\gamma,T,E^{0}_{i}) = 0 \qquad (20)$$

if T does not include R. Lemma 1 and Axiom 3 lead to the statement: for all $v\in V$, $\sum_{i\in R} [\Phi^{0}_{i}(v,\bar{\gamma})-\Phi_{i}(v,\bar{\gamma})] = 0$. But this, with the help of inductive assumption, (3), (11), (10) and (20), can be equivalently rewritten as

$$\sum_{i\in R} [\rho_{i}(\bar{\gamma},R\setminus i,E^{0}_{i})-\rho_{i}(\bar{\gamma},R\setminus i,E_{i})]\bar{\gamma}^{R\setminus i}_{i}[v(R)-v(R\setminus i)] = 0$$

for all $v\in V$. Therefore, it follows by (10), that

$$\sum_{\rho\in\Lambda(R\setminus i)} [E^{0}_{i}(\rho)-E_{i}(\rho)]\gamma(\rho(x^{R\setminus i}_{1}),x^{R\setminus i}_{1})\gamma(S_{2},x^{R\setminus i}_{2})\times\cdots\times\gamma(\rho(x^{R\setminus i}_{k}),x^{R\setminus i}_{k})=0$$

for $i\in R$ and for all R-exhausting $\gamma\in\Gamma$. But here we can apply Lemma 6 to get immediately that $E_i(\rho) - E_i^{\circ}(\rho) = 0$ for all $i\in R$ and $\rho\in\Lambda(R\setminus i)$. Now taking into consideration the fact that R can be chosen arbitrarily, and $|R\setminus i|=k$, we immediately get the validity of (19) under $|T|=k$. Thus, by induction principle, the lemma has been proved.

Proof of Theorem 1. The result is an immediate consequence of Lemmata 1, 2, 5 and 7.

III. <u>How to measure the coefficients of initiative and attraction</u>

Suppose we are given a fixed set of individuals $N = \{1,\cdots,n\}$ characterized by some fixed but unknown indicators of initiative and attraction γ^*. To find out the most adequate evaluation $\bar{\gamma}$ for γ^* we let the individuals play a series of characteristic function games and then we may either try to find the values of an evaluation for γ^* directly, without referring to our theory of initiative and attraction, or otherwise we may try to find them by comparing the actual outcomes of those games with formulae expressing the value and involving γ. Below, we shall briefly describe two approaches of the first kind and five approaches of the second kind. The first approach is only available in the case where we have a complete information about the actual coalition formation process; the second, based on the analysis of empirical data, may produce untrue results but still it gives some information about the frequency of players' participation in decisive coalitions. The last five approaches are related to various mathematical procedures.

It should be stressed, however, that the actual initiative and attraction of individuals and coalitions may heavily depend on which games are actually being played. What we are looking for and expecting to determine is a sort of "absolute" or "ideal" indicators of initiative and attraction, independent of the prevailing circumstances and interests. As a consequence, the results obtained by Approaches III-VII may considerably differ from the observed behavior of individuals and coalitions and described by means of empirical Approaches I-II.

Formally, suppose that the individuals have played games $v^1,\cdots,v^k$ and, in a game v^j, they decided to distribute the total revenue of $v^j(N)$, or a part of it, according to a payoff vector $(\alpha_1^j,\cdots,\alpha_n^j)$.

Approach I (detection or players' declaration). Suppose that, in the case of some games v^j, the outcome vector $(\alpha_1^j,\cdots,\alpha_n^j)$ has been reached in the following way: a coalition S has formed in some order $(i_1,\cdots i_{|S|})$ and decided to distribute the total revenue according to a payoff vector $(\alpha_1^j,\cdots,\alpha_n^j)$ while the remaining players were not objecting against it. Let J be the set of all indexes of such games. We may try to detect, either by asking the individuals involved directly, or by some other available means, in the case of each game, what was that coalition and in which order it has formed. Suppose that, in a game v^j, there was such a coalition and that it has formed in some order $(i_1^j,\cdots,i_{m_j}^j)$. The reasonable assignment of indicators of initiative and attraction in this case seems to be the following: for any coalition S denote

$$A:=\left|\{i\in J \mid S=\{i_1^j,\cdots,i_{|S|}^j\}\}\right|$$

and, for any S and $i\in S$, define

$$\bar{\gamma}_i^S := \begin{cases} \left|\{i\in J \mid S=\{i_1^j,\cdots,i_{|S|}^j\} \text{ while } i=i_{|S|+1}^j\}\right|\cdot A_S^{-1} & \text{if } A_S\neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

A disadvantage of this approach is that often no information about the actual coalition forming process is available.

Approach II (uniform average). For a game v^j, denote by S^j the set $\{i\in N \mid \alpha_i^j > v(\{i\})\}$. We are assuming that S^j is the coalition that has actually formed and decided about the payoff for v^j. (We restrict the attention just to the case where there are no such disjoint coalitions). We are also assuming that all orderings of players in which this coalition has formed are equally likely. Thus we define, for any $i=1,\cdots,k$, any coalition S and any $i\in S$,

$$\alpha_S^{ji} := \begin{cases} |S^j\setminus S|^{-1} & \text{if } i\in S^j\setminus S, \\ 0 & \text{otherwise} \end{cases}$$

and, for any coalition S and any $i \in S$,

$$\bar{\gamma}_i^S := \Big[\sum_{j:S\subset S^j} |S^j|!^{-1}\Big]^{-1} \cdot \sum_{j:S\subset S^j} \alpha_S^{ji} |S^j|!^{-1}.$$

Approach III (solving system of equations). A system of indicators γ_i^S seems to fit best the data if the following system of equations and inequalities is satisfied:

$$\sum_{T\subseteq N\setminus i} q(\gamma,T)\gamma_i^T[v^j(T\cup i)-v^j(T)] = \alpha_i^j, \quad \text{for } j=1,\dots,k,\ i=1,\dots,n;$$

$$\sum_{i\notin T} \gamma_i^T \le 1, \quad \text{for } T\subset N; \tag{21}$$

$$0 \le \gamma_i^T, \quad \text{for } T\subset N,\ i\notin T.$$

The proposed method suggests solving this system for γ (notice that $q(\gamma,T)$ is just an abbreviation for a polynomial with variables γ_i^T). A disadvantage of this method is that (21) has usually no solution, especially if the number of games played is large (the number of variables is constant). If this is the case, we turn to Approach III in which we do not attempt to solve (21) but only try to minimize the error.

Approach IV (error minimization). Assuming the least power (e.g. square) optimization rule, we are trying to find indicators γ_i^T minimizing the value of the expression ($p\ge 1$)

$$E_p(\gamma) = \sum_{\substack{i=1,\dots,n \\ j=1,\dots,k}} \Big|\sum_{T\subseteq N\setminus i} q(\gamma,T)\gamma_i^T[v^j(T\cup i)-v^j(T)] - \alpha_i^j\Big|^p$$

subject to the constraints

$$\sum_{i\notin T} \gamma_i^T \le 1, \quad \text{for } T\subset N; \tag{22}$$

$$0 \le \gamma_i^T, \quad \text{for } T\subset N,\ i\notin T.$$

Note that the minimized objective function is a polynomial of degree $\le pn$ while the constraints are linear.

Alternatively, we may be interested in minimizing the expresssion

$$E_o(\gamma) = \max_{\substack{i=1,\dots,n \\ j=1,\dots,k}} \left|\sum_{T\subseteq N\setminus i} q(\gamma,T)\gamma_i^T[v^j(T\cup i)-v^j(T)] - \alpha_i^j\right|$$

subject to the constraints

$$\sum_{i\in T} \gamma_i^T \le 1, \quad \text{for } T\subset N; \tag{23}$$

$$0 \le \gamma_i^T, \quad \text{for } T\subset N,\ i\in T.$$

A disadvantage of this method is that it may lead to a large variety of distinct and remote solutions. A considerable improvement obtains after a modification described below.

Approach V (average of near optimal solutions). Under this approach we calculate, for a fixed $\rho\ge1$ or $\rho\ge0$ and $\varepsilon\ge0$, an (integral) average of (all) optimal or ε-optimal solutions to (22) or (23). A numerical procedure to do so may require determination of the minimal value E_p for $E_p(\gamma)$ (e.g. by an optimization procedure or by generating at random a large number of systems of indicators γ and taking the minimal error $E_p(\gamma)$).
Then one may generate a reasonably large number ρ of systems of indicators γ and take the average of all those γ of them for which $E_p(\gamma)\le E_p+\varepsilon$.

An advantage of this approach is its numerical simplicity and uniqueness of the obtained result.

Approach VI (weighted average). Let, for a system of indicators of initiative and attraction γ and a positive integer ρ, $E_p(\gamma)$ have the same meaning as in the Approach V; assume that for some $\delta>0$ and all γ, $E_p(\gamma)>\delta$. For a positive integer r, the following $|A|$-dimensional integral expresssion

$$\bar{\gamma} = \int\gamma E_p^{-r}(\gamma)d\gamma\cdot(\int E_p^{-r}(\gamma)d\gamma)^{-1}$$

(the first integral is taken with respect to the $|A|$-dimensional while the second with respect to 1-dimensional Lebesgue measure) can be regarded as an evaluation of the true indicatores γ^*, since it is actually a weighted average of all γ with weights inversely proportional to the error corresponding to a given γ.

An advantage of this method is its numerical simplicity (for purposes of the present paper we have been computing the integral by means of a random procedure and we have combined it with calculations for the Approach V) and uniqueness of the result. A disadvantage is that, for small values of ρ and r it gives a solution near the center of gravity of Γ, otherwise we are faced with error accumulation problems.

Approach VII (statistical). Applying this approach, one is assuming a statistical hypothesis concerning the distribution of the indicators γ and test it with respect to the obtained data α_i^j. A disadvantage of this method is that one can hardly assume that the distributions of different γ_j^T are independent and even if this were the case, the distrbutions of the involved random variables $\varphi(\gamma,T)$ are very difficult to determine. This approach has not been used while dealing with special case presented in Section IV.

IV. <u>An example: experimental determination of initiative and attraction of 3 individuals</u>

The following experiment has been performed at Warsaw University in April 1991: three students, named 1, 2 and 3 have been asked to play the following games:

Game \ Coalition	123	12	13	23	1	2	3	∅
υ^1, υ^2	1	1	1	1	0	0	0	0
υ^3, υ^4	1	0.8	0.8	0.8	0	0	0	0
υ^5	1	1	1	0	0	0	0	0
υ^6	1	1	0	1	0	0	0	0
υ^7	1	0	1	1	0	0	0	0
υ^8	1	0.8	0.8	0	0	0	0	0
υ^9	1	0.8	0	0.8	0	0	0	0
υ^{10}	1	0	0.8	0.8	0	0	0	0

Table 1. Description of games υ^1 thru υ^{10}

The Players have been situated in separate rooms and they were able to negotiate, pairwise, by phone. In each game, after an agreement (division of the payoff among the Players) has been reached, it was announced to the Referee. The Players have decided for the following payoffs:

Player\Game	1	2	3	4	5	6	7	8	9	10	TOTAL
1	0	0.5	0.4	0.4	0.82	0.18	0	0.7	0	0.15	3.15
2	0.5	0.5	0	0.4	0.18	0.82	0.16	0.15	0.67	0	3.38
3	0.5	0	0.4	0	0	0	0.84	0.15	0.13	0.65	2.67
TOTAL	1	1	0.8	0.8	1	1	1	1	0.8	0.8	9.2

Table 2. Outcomes reached in the experiment

Note that the payoffs obtained in Games 3, 4, 9 and 10 are inefficient. It may be of interest to compare the actual outcomes with Shapley values of the respective games, as shown in Table 3 below:

Player \ Game	1,2,3,4	5	6	7	8	9	10
1	0.333	0.667	0.167	0.167	0.6	0.2	0.2
2	0.333	0.167	0.667	0.167	0.2	0.6	0.2
3	0.333	0.167	0.167	0.667	0.2	0.2	0.6

Table 3. The Shapley value of games v^1 thru v^{10}

Using the data obtained in the course of the experiment, the values of the indicators of initiative and attraction have been determined (or we tried to do so) by applying Approaches I thru VI. The results are presented below in Table 4. The evaluation of coefficients γ^S_i for $\#S\geq 2$ are omitted since forming the grand coalition N, in most cases, has been strategically useless and therefore the available data are somewhat unreliable. To obtain the data in case of Approach I, the participants in the experiment have been asked about the actual process of forming the succesfull coalition.The minimal errors have been found E_2= 0.779, E_1=3.670. All random procedures in Approach VI were involving 10,000 random samples; those in Approach V - 400 samples;

usually computations have been performed several times and leaded to the same or nearly the same results.
The results obtained by the Approach V seem to be most adequate and they do not differ much for different values of the parameters ρ and ε involved.

	γ_1^0	γ_2^0	γ_3^0	γ_1^2	γ_1^3	γ_2^1	γ_2^3	γ_3^1	γ_3^2
Appr.I	0.2	0.5	0.3	0.6	0.667	0.5	0.333	0.5	0.4
Appr.II	.3333	.3833	.2833	.5625	.4167	.6429	.5833	.3571	.4375
Appr.III	System of equations (21) has no solution								
Appr.IV	.3291	.3397	.3313	.5760	.4244	.7138	.5756	.2862	.4240
Appr.V									
$\rho=$ $\varepsilon=$									
2 .08	.362	.346	.289	.619	.557	.790	.433	.200	.369
2 .12	.372	.350	.275	.628	.555	.776	.438	.213	.360
2 .18	.386	.355	.255	.684	.555	.737	.429	.247	.333
2 .22	.395	.367	.235	.647	.577	.710	.408	.271	.328
1 .18	.361	.303	.336	.911	.547	.918	.449	.078	.085
1 .33	.360	.321	.317	.872	.540	.906	.453	.085	.120
Appr.VI									
$\rho=$ $r=$									
2 1	.259	.256	.263	.344	.339	.346	.346	.335	.335
2 2	.269	.257	.275	.353	.345	.362	.362	.334	.340
2 4	.280	.269	.296	.373	.353	.395	.390	.342	.351
1 1	.253	.252	.256	.338	.338	.341	.337	.334	.339
1 2	.260	.250	.261	.345	.340	.350	.342	.333	.326
1 4	.269	.255	.271	.359	.342	.369	.351	.324	.330
0 1	.255	.253	.251	.337	.339	.337	.336	.335	.338
0 2	.259	.258	.255	.340	.340	.340	.342	.342	.340
0 4	.264	.266	.261	.349	.348	.354	.349	.350	.351
4 4	.305	.302	.307	.426	.360	.449	.460	.366	.460
6 6	.263	.387	.268	.399	.282	.406	.628	.410	.463

Table 4. The indicators of initiative and atraction for Players 1-3 in the experiment, obtained in different approaches. In the case of the approach IV we have taken ρ=2.

V. Discussion of the literature

The concept of value of games with indicators of initiative and attraction generalizes one known as Owen's value [1977] (cf. also Carreras and Owen [1988]).

Given a fixed set of players $N = \{1,2,\cdots,n\}$, the set V of all power functions over N and the set β of all partitions (coalition structures) of N, the Owen's value is defined as a function $\Psi: V\times\beta \longrightarrow \mathbb{R}^n$ satisfying the usual axioms of efficiency, symmetry and additivity and, additionally, the following:

If the game v among coalitions $(B_1,\cdots,B_k)$ forming a coalition structure $\mathcal{B}$ is inessential then, for each $j = 1,\cdots,k$, $\Sigma_{i\in B_j}\Psi_i(v,\mathcal{B}) = v(B_i)$ (cf. Kurz [1988]).

It is not difficult to check that $\Psi(v,\mathcal{B}) = \Phi(v,\gamma)$ for γ defined as follows:

$$\gamma(T,i) := \begin{cases} |N\setminus T|^{-1} & \text{if } T \text{ is a union of some } B_l\text{'s;} \\ |B_j\setminus T|^{-1} & \text{if } T \text{ is a union of some } B_l\text{'s and a part of } B_j \text{ with } i\in B_j; \\ 0 & \text{otherwise.} \end{cases}$$

This system γ has a natural interpretation: the members of a coalition B_j in a coalition structure $\mathcal{B}$ join an already existing coalition T either immediately, i.e. in some order not interrupted by any members of any remaining B_l's or they do not join T at all. This means that the members of the remaining B_l's do not even have time to think of joining T before all members of B_j do so; this means that in this case $\gamma(T,i) = 0$ whenever $i\in B_l$ for $l\neq j$.

The results in the present paper are also related to those of Kalai and Samet [1987] or Weber [1988], the main difference being that those authors rather deal with just one value (in our terminology it would be $\Phi(\cdot,\gamma_0)$) while we consider simultaneously the whole family $(\Phi(\cdot,\gamma)|\gamma\in\Gamma)$.

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