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Abstract • Содержание • Streszczenie

In this paper we introduce the notion of a rough relations, which is based on the idea of a rough set defined by the author in the previous paper. The notions of a rough set and the rough relations can be considered as an alternative to fuzzy set and relation.

Some elementary properties of rough relations are listed.

Приближенные отношения

В работе вводится понятие приближенных отношений, учитывающее введенное ранее автором понятие приближенного множества. Понятие приближенного множества и приближенного отношения можно считать альтернативными предложениями в отношении размытых множеств. Оговариваются элементарные свойства приближенных отношений.

Relacje przybliżone

W artykule wprowadzamy pojęcie relacji przybliżonych, które jest oparte na pojęciu zbioru przybliżonego zdefiniowanego przez autora w poprzedniej pracy. Pojęcia przybliżonych zbiorów i przybliżonych relacji mogą być rozpatrywane jako alternatywa rozmytych zbiorów i relacji.

Niektóre podstawowe własności relacji przybliżonych są wymienione.

1. INTRODUCTION

In [1] we introduced the notion of a rough set used to define approximate operations on sets. The notion of a rough set can be viewed as an alternative to fuzzy set, however there are some essential differences between these two notions.

In this note we define the notion of a rough relation, which is based on the previously defined notion of a rough set. Basic idea of the notion of rough relation is connected with the fact that in some applications we are unable to say for sure whether some pair (x,y) belongs to the relation $R \subset X \times Y$ or not. To deal with this kind of situations the introduced notions seem to be of some value.

Some elementary properties of rough relations are stated at the end of this paper.

2. ROUGH SETS

Before we define the notion of a rough relation first we recall the notion of a rough set after [1]; which will be used as a departure point of our considerations.

Let U be some set called <u>universum</u> and let R be an equivalence relation on U, called here <u>indiscernity relation</u>. The pair $A = \langle U,R \rangle$ will be called an <u>approximation space</u>. Equivalence classes of the indiscernity relation R will be referred to as <u>elementary sets</u> in A, and every union of elementary sets in A will be called a <u>composed set</u> of A.

If $X \subset \mathcal{U}$, then the least composed set in A containing X will be called the <u>best upper approximation</u> of X in A (denoted as $\bar{A}X$), and the greatest composed set in A contained in X will be called the best lower approximation of X in A (denoted as $\underline{A}X$).

For every approximation space and every set $X \subset U$, the following axioms are true:

A1. $\bar{A}X \supset X \supset \underline{A}X$,

A2. $\bar{A}1 = \underline{A}1 = 1$,

A3. $\bar{A}0 = \underline{A}0 = 0$,

A4. $\bar{A}\bar{A}X = \underline{A}\bar{A}X = \bar{A}X$,

A5. $\underline{A}\underline{A}X = \bar{A}\underline{A}X = \underline{A}X$,

A6. $\bar{A}(X \cup Y) = \bar{A}X \cup \bar{A}Y$,

A7. $\underline{A}(X \cap Y) = \underline{A}X \cap \underline{A}Y$,

A8. $\bar{A}X = -\underline{A}(-X)$,

A9. $\underline{A}X = -\bar{A}(-X)$.

We use here 0,1 as an abbreviation for the empty set and the universum respectively and $-X$ is denoting $1-X$.

Moreover we have

1) $\bar{A}(X \cap Y) \subset \bar{A}X \cap \bar{A}Y$,

2) $\underline{A}(X \cup Y) \supset \underline{A}X \cup \underline{A}Y$,

3) $\bar{A}X - \bar{A}Y \subset \bar{A}(X-Y)$,

4) $\underline{A}X - \underline{A}Y \supset \underline{A}(X-Y)$.

We introduce three kinds of rough equality of sets. Let $X,Y \subset U$, then we have:

a) the sets X,Y are roughly bottom equal in A, in symbols $X \underset{A}{\eqsim} Y$, iff $\underline{A}X = \underline{A}Y$,

b) the sets X,Y are roughly top equal in A, in symbols $X \underset{A}{\simeq} Y$, iff $\overline{A}X = \overline{A}Y$,

c) the sets X,Y are roughly equal in A, in symbols $X \underset{A}{\approx} Y$, iff $X \underset{A}{\simeq} Y$ and $X \underset{A}{\eqsim} Y$.

It is easy to show that $\eqsim$, $\simeq$, $\approx$ are equivalence relations on $P(U)$.

We introduce three kinds of rough subsets. Let $X,Y \subset U$, then we shall say

i) the set X is a rough bottom subset of Y in A in symbols $X \underset{A}{\underset{\sim}{\subset}} Y$ iff $\underline{A}X \subset \underline{A}Y$.

ii) the set X is a rough top subset of Y in A, in symbols $X \underset{A}{\overset{\sim}{\subset}} Y$, iff $\overline{A}X \subset \overline{A}Y$,

iii) the set X is a rough subset of Y in A, in symbols $X \underset{A}{\overset{\sim}{\underset{\sim}{\subset}}} Y$, iff $X \underset{A}{\underset{\sim}{\subset}} Y$ and $X \underset{A}{\overset{\sim}{\subset}} Y$.

One can easily show that $\underset{\sim}{\subset}$, $\overset{\sim}{\subset}$, $\overset{\sim}{\underset{\sim}{\subset}}$ are ordering relations.

3. ROUGH RELATIONS

Let $B = \langle X,R \rangle$ and $C = \langle Y,Q \rangle$ be two approximation spaces. By a product of B and C we shall mean the space $A = \langle Z,P \rangle$, denoted $A = B\times C$, where $Z = X\times Y$ and the indiscernity relation $P \subset (X\times Y)^2$ defined as follows:
$P((x_1,y_1),(x_2,y_2))$ iff $R(x_1,x_2)$ and $Q(y_1,y_2)$ where $x_1,x_2 \in X$ and $y_1,y_2 \in Y$.

Of course P is an equivalence relation and every equivalence class of P is of the form $e \times e'$, where e, e' are some equivalence classes of the relations R and Q respectively.

Any equivalence class of the indiscernity relation P will be called an elementary relation in A. Union of elementary relations in A will be called a compared relation in A.

Let $A = B \times C$, where $B = \langle X, R \rangle$ and $C = \langle Y, Q \rangle$, and let $S \subset X \times Y$ be some binary relation on $X \times Y$.

By the best upper approximation of S in A (denoted $\bar{A}S$) we shall mean the least composed relation in A containing S.

By the best lower approximation of S in A (denoted $\underline{A}S$) we shall mean the greatest composed relation in A contained in S.

Because rough relations are some kind of rough sets properties AI - A9 and 1) - 4) given in the previous paragraph are also valid for rough relations.

Also rough equality of relations and rough inclusion of relations are defined exactly in the same way as for rough sets.

We can also easily extend the definition of rough relation for any number of arguments.

4. SOME PROPERTIES OF ROUGH RELATIONS

Let $A = B \times B$, be a product of approximation spaces, where $B = \langle X, R \rangle$ and let $Q \subset X^2$ be some binary relation on X.

One can easily verify the following properties:

1) If Q is an identity relation then neither $\underline{A}Q$ nor $\overline{A}Q$ is an identity relation; $\overline{A}Q$ is a symmetric relation and $\underline{A}Q = \emptyset$.

2) If Q is a reflexive relation then $\overline{A}Q$ is reflexive to, but $\underline{A}Q$ is not reflexive

3) If Q is a symmetric relation so are $\underline{A}Q$, and $\overline{A}Q$ (provided $\underline{A}Q \neq \emptyset$)

4) If Q is an antysymmetric relation so are $\underline{A}Q$ and $\overline{A}Q$.

5) If Q is a nonsymmetric relation so are $\underline{A}Q$ and $\overline{A}Q$.

6) If Q is a transitive relation then neither $\underline{A}Q$ nor $\overline{A}Q$ is transitive.

7) If Q is an equivalence relation, then $\overline{A}Q$ is a tolerance relation (reflexive and symmetric) and $\underline{A}Q$ defines a cover of the set $X = X_1 \cup X_2 \cup \ldots \cup X_n$, $n \geqslant 1$, where $X_i = \overline{B}Z_i$, and Z_i is an equivalence class of Q, i.e. $X = Z_1 \cup Z_2 \cup \ldots \cup Z_n$, $Z_i \cap Z_j \neq \emptyset$, for each $i \neq j$, $i,j = 1,2,\ldots,n$.

8) If Q is an equivalence relation, then $\underline{A}Q$ is also an equivalence relation on some set $Y \subset X$, and its equivalence classes are of the form $\underline{B}Z_i$, where Z_i is an equivalence class of Q, i.e. $X = Z_1 \cup Z_2 \cup \ldots \cup Z_n$, $Z_i \cap Z_j \neq \emptyset$, for $i \neq j$, $i,j = 1,2,\ldots,n$.

9) If Q is an ordering relation then neither $\underline{A}Q$ nor $\overline{A}Q$ is an ordering relations.

10) $\underline{A}(Q^{-1}) = (\underline{A}Q)^{-1}$
$\overline{A}(Q^{-1}) = (\overline{A}Q)^{-1}$,

11) If $Q = S \circ T$ (Q is a composition of S and T), then

$$\underline{A}Q = \underline{A}S \circ \underline{A}T,$$

$$\overline{A}Q = \overline{A}S \circ \overline{A}T,$$

12) If $Q^{*} = \bigcup_{i=0}^{\infty} Q^{i}$ (Q^{*} is a transitive closure of Q) then

$$\underline{A}Q^{*} = \bigcup_{i=0}^{\infty} (\underline{A}Q)^{i},$$

$$\overline{A}Q^{*} = \bigcup_{i=0}^{\infty} (\overline{A}Q)^{i}.$$

REFERENCES

1 Z. Pawlak, Rough Sets, Basic Notions, ICS PAS Reports, 431, 1981.