# ON THE INTEGRATION OF EQUATIONS OF CELESTIAL MECHANICS USING DIGITAL COMPUTERS* 

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The basic problem of celestial mechanics - the motion of celestial bodies - is characterized by a system of ordinary differential equations of the second order in which no first derivatives and an explicit independent variable occur:

$$
\begin{equation*}
\ddot{x}_{i}=f_{i}\left(x_{1}, x_{2}, \ldots, x_{n}\right) \quad(i=1,2, \ldots, n) \tag{1}
\end{equation*}
$$

where $n$ is a number of the equations. (The point above the symbol of the variable means differentiation with respect to time). In computation practice, the most widespread method of solving such a system of equations is numerical integration. The use of very fast digital computers gives new possibilities of applying various methods of approximate integration of differential equations.

This paper is to present several criteria relative to the methods of Runge-Kutta-Gill, de Vogelaere, Adams-Störmer and Gautschi, which enable the choice of the integration method and the proper step of integration, best adapted for the numerical solution of the given problem of celestial mechanics.

In Part I of this paper the integral formulae of the methods under consideration, the flow charts according to which the computation program is prepared, and a brief description of each method, with respect to its application to digital computers, are presented.

Part II analyses the way of carrying out the task outlined in this paper ( $\S 5$ and $\S 6$ ) and sums up the results thus obtained ( $\S 7, \S 8$ and $\S 9$ ).

## Part I

§1. The Runge-Kutta-Gill method. When solving second order equations of type (1) with $n=3$ by the Runge-Kutta-Gill method [3]

[^0]it is necessary to transform them into the equivalent system of seven equations of the first order:
\[

$$
\begin{aligned}
& \dot{z}_{1}=1 \\
& \dot{z}_{2}=f_{1} \\
& \dot{z}_{3}=f_{2} \\
& \dot{z}_{4}=f_{3} \\
& \dot{x}_{1}=z_{2} \\
& \dot{x}_{2}=z_{3} \\
& \dot{x}_{3}=z_{4}
\end{aligned}
$$
\]

or generally:

$$
\dot{y}_{i}=g_{i}\left(y_{1}, y_{2}, \ldots, y_{7}\right) \quad(i=1,2, \ldots, 7)
$$

where $g_{1}(t) \equiv 1$. Using the fourth order method we introduce the following notation:

$$
\begin{gathered}
\dot{y}_{i j}=k_{i j}=g_{i}\left(y_{1, j-1}, y_{2, j-1}, \ldots, y_{7, j-1}\right)=f_{i, j-1} \\
(i=1,2, \ldots, 7 ; \quad j=1,2,3,4)
\end{gathered}
$$

The general formula for the value of the sought function in the $m$-th step of integration, when the $(m-1)$-th step is known is:

$$
\begin{gathered}
y_{i j}=y_{i, j-1}+h\left[a_{j}\left(k_{i j}-b_{j} q_{i, j-1}\right)\right] \\
(i=1,2, \ldots, 7 ; \quad j=1,2,3,4)
\end{gathered}
$$

where

$$
q_{i j}=q_{i, j-1}+3\left[a_{j}\left(k_{i j}-b_{j} q_{i, j-1}\right)\right]-o_{j} k_{i j}
$$

and

$$
\begin{array}{lll}
a_{1}=\frac{1}{2} & b_{1}=2 & c_{1}=\frac{1}{2} \\
a_{3}=1-\sqrt{\frac{1}{2}} & b_{2}=1 & c_{2}=1-\sqrt{\frac{1}{2}} \\
a_{3}=1+\sqrt{\frac{1}{2}} & b_{3}=1 & c_{3}=1+\sqrt{\frac{1}{2}} \\
a_{4}=\frac{1}{6} & b_{4}=2 & c_{4}=\frac{1}{2}
\end{array}
$$

putting

$$
\begin{aligned}
q_{i 1}\left(t_{1}\right) & =0 \\
q_{i 1}\left(t_{m}\right) & =q_{i n}\left(t_{m-1}\right) \quad(m=r, 2, \ldots) \\
t_{m} & =t_{1}+(m-1) h
\end{aligned}
$$

In the above given formulae and in the following text $h$ stands for the step of integration.

One can accomplish a computation program according to the following flow chart in which, as in the next ones, both input and output are designated by a circle and a rectangle is used to designate all the operations except logical criterions of the choice of a proper way, the blocks of which are designated by an oval.


The Runge-Kutta-Gill method of the fourth order requires $3(2 n+1)$ cells to store the quantities $k_{i j}, y_{i j}, q_{i j}$ and twelve cells to store the constants $a_{j}, b_{j}, c_{j}$. (Although some constants are repeated the organisation of the program requires placing them in separate cells.) Hence $6 n+15$ cells in operational memory of digital computer should be reserved. One starting point is needed to begin computations.
§ 2. The de Vogelaere method. The de Vogelaere method [5] is used for ordinary differential equations of the second order in which no first derivatives occur. Using the fourth order method, the following subsidiary quantities are calculated in the first step of integration:

$$
\begin{gathered}
X_{i 1}=\frac{1}{2} h \dot{x}_{i 1} \\
x_{i, 1 / 2}=x_{i 1}-\frac{1}{2} h \dot{x}_{i 1}+\frac{1}{8} h^{2} f_{i 1} \quad(i=1,2,3)
\end{gathered}
$$

The value of the sought function in each next $m$-th step of integration, when ( $m-1$ )-th step is known, is derived from the formula:

$$
\begin{aligned}
x_{i, m}= & x_{i, m-1}+2 X_{i, m-1}+2 F_{i, m-1}+F_{i, m-1 / 2} \\
& (i=1,2,3 ; \quad m=1,2, \ldots)
\end{aligned}
$$

where

$$
\begin{gathered}
F_{i, m}=\frac{1}{12} h^{2} f_{i, m} \quad F_{i, m-1 / 2}=\frac{1}{3} h^{2} f_{i, m-1 / 2} \\
x_{i, m-1 / 2}=x_{i, m-1}+X_{i, m-1}-\frac{1}{8} F_{i, m-3 / 2}^{\prime}+2 F_{i, m-1} \\
X_{i, m}=X_{i, m-1}+F_{i, m-1}+F_{i, m-1 / 2}+F_{i, m} \\
(i=1,2,3 ; \quad m=1,2, \ldots)
\end{gathered}
$$

The de Vogelaere method requires reservation of $4 n$ cells in the operational memory of digital computer. These cells are designated by $P$, $Q, R, S$ (the index is omitted for simplicity sake). The contents of these cells are designated $(P),(Q),(R),(S)$ respectively.

Let

$$
(P)=x_{i, m-1}, \quad(Q)=X_{i, m-1}, \quad(R)=F_{i, m-1}, \quad(S)=F_{i, m-3 / 2}
$$

The computations in one integration step are carried out according to the flow chart (see p. 49).

Three decimal constants occurring in the formulae of this method are to be stored in the operational memory of the computer. Hence $4 n+3$ cells in operational memory should be reserved, when the de Vogelaere method is used. One starting point is needed, as in the case of the previous method.
§ 3. The Adams-Störmer method. The formulae of the difference method of Adams-Störmer were given by Kamke [4]. They are the following:
I. $x_{i, m+1}=x_{i, m}+x_{i, m-4}-x_{i, m-s}+$

$$
\begin{gathered}
+\frac{h^{2}}{48}\left(67 f_{i, m}-8 f_{i, m-1}+122 f_{i, m-2}-8 f_{i, m-3}+67 f_{i, m-4}\right) \\
\text { II. } x_{i, m+1}=x_{i, m}+x_{i, m-2}-x_{i, m-3}+ \\
+\frac{h^{2}}{240}\left(17 f_{i, m+1}+232 f_{i, m}+222 f_{i, m-1}+232 f_{i, m-2}+17 f_{i, m-3}\right) \\
(i=1,2,3 ; \quad m=1,2, \ldots)
\end{gathered}
$$

The flow chart of the computations in one step of integration is the following:

Computatio - 4


Flow chart of the de Vogelaere method


Flow chart of the Adams-Störmer method

The sought functions (and the corresponding right sides of the equations) are calculated in the first approximation from the formulae of group I when $s=0$ and next, when $s=1$, more exact formulae (of group II) are used consecutively for all $n$ equations.

Transition to computations in each next step of integration requires the proper shiftings of the contents of the cells $x$ and $f$ according to the flow chart in the box marked by an asterisk.

The method under consideration requires $12 n$ cells for storing the values of functions and right sides of the equations and 10 cells for ten constants occurring in the formulae. (As in the case of the Runge-Kutta-Gill method, though some constants are repeated, they are to be placed in separate cells for the sake of the organisation of the program.)

Hence $12 n+10$ cells in operational memory should be reserved. Six starting points are to be known to begin computations.
§ 4. The Gautschi method. For equations having periodical solutions Gautschi [2] worked out a special method taking advantage of the quadrature formulas based on trigonometric extrapolation polynomials.

For the equations of type (1) the values of the sought functions in each step of integration are derived from the formulae:

$$
\begin{gathered}
x_{i, m+1}+a_{p, 1}(v) x_{i, m}+a_{p, 2}(v) x_{i, m-1}=h^{2} \sum_{i=1}^{2 p-1} \beta_{p, \lambda}(v) f_{i, m+1-\lambda} \\
(i=1,2,3 ; \quad m=1,2, \ldots)
\end{gathered}
$$

$p=2$ corresponds to the fourth order method; the parameter $v$ is expressed by the formula:

$$
v=h \frac{2 \pi}{T}
$$

where $T$ is the period of solution. The constants $\alpha$ and $\beta$ with the appropriate indices have the form:

$$
\begin{aligned}
& \alpha_{21}=-2\left(1-\frac{1}{6} v^{4}+\frac{1}{36} v^{6}+\ldots\right) \\
& \alpha_{22}=-\alpha_{21}-1 \\
& \beta_{21}=\frac{13}{12}\left(1-\frac{19}{52} v^{2}+\frac{7}{120} v^{4}+\ldots\right) \\
& \beta_{22}=-\frac{2}{12}\left(1-\frac{9}{4} v^{2}+\frac{37}{120} v^{4}+\ldots\right) \\
& \beta_{21}=\frac{1}{12}\left(1+\frac{1}{4} v^{2}+\frac{7}{120} v^{4}+\ldots\right)
\end{aligned}
$$

The program of computations which are to be carried out in one step of integration can be set up according to the flow chart:


Before the first step of integration is carried out the subsidiary quantities $\alpha$ and $\beta$ with the corresponding indices must be computed. The Gaustchi method equivalent to the fourth order methods ( $p=2$ ) requires the reservation of five storage cells for these constants; $7 n$ storage cells must be reserved for functions and right sides of equations. Hence $7 n+5$ cells of digital computer operational memory should be set apart. Three starting points have to be known to begin computations.

## Part II

§5. To establish criteria permitting the choice of one of the numerical integration methods discussed in Part I, to solve type (1) equations by digital computers, we have to find the value of the total error of the solution obtained by an approximate method for every step of integration. There are three causes of such error:

1. Assumption of a finite number of terms in integral formulae;
2. Inaccuracy of input data;
3. Rounding off in every step of integration.

The value of this error can be determined when both the approximate and the exact solution of a given equation are known. Hence in order to analyse the error for the various methods we must choose a set of equa-
tions which is representative, as much as possible, of the equations of motion of celestial mechanics and having an exact analytical solution.

These two conditions are satisfied by, e. g., the equations of elliptical motion of two point bodies (e. g., the motion of the Moon about the Earth, disturbed by the gravitational activity of the Sun) occurs most frequently in practice, its equations cannot be taken as an example because its exact solutions are not known. These equations differ from the equations of motion of two bodies only by a small factor in right sides, called perturbation. It causes small deformations of the elliptical trajectory. This trajectory is just determined by the equations of motion of two material points, that is Kepler motion.

Since these disturbances may be neglected we may extend the qualitative results concerned with the analysis of errors of numerical integration of the Kepler motion equations so as to cover the motions affected by the perturbation force of a third body.

The second reason for choosing the system of the equations of motion of two bodies is the existance of its exact analytical solutions, given by Kepler [1].

If the mass of the first body is taken as a unit and the mass of the second body which is small in comparison with the first one (e. g. the Sun and a planet moving about it) is disregarded and the Cartesian coordinate system with the origin at the point in which the body having the greater mass occurs is adopted, the motion of the smaller body is described by the following system of equations:

$$
\begin{equation*}
\ddot{x}_{i}=-k^{2} \frac{x_{i}}{r^{3}} \quad(i=1,2,3) \tag{2}
\end{equation*}
$$

where the distance between the bodies $r=\sqrt{\sum_{i} x_{i}^{2}}$, and $k$ is the coefficient known as the Gauss gravitational constant.

Kepler's solution of these equations has the form:

$$
\begin{equation*}
x_{i}=a\left[P_{i}(\cos E-e)+Q_{i} \sqrt{1-e^{2}} \sin E\right] \quad(i=1,2,3) \tag{3}
\end{equation*}
$$

where $E$ - called the eccentric anomaly - is a function of time and is given by Kepler's equation:

$$
E-e \sin E=\mu t
$$

The constants $a$ and $e$ denote the semi-major axis and eccentricity of the ellipse respectively, that is two out of the six orbital elements, characterising its shape. The coefficient $\mu$ occurring at the right side of Kepler's equation is called the mean motion; it is connected with the semi-major axis of the orbit by the simple formula:

$$
\mu=k a^{-3 / 2}
$$

The constants $P_{i}$ and $Q_{i}$ are expressed by three other orbital elements determining the position of the orbit is space: $\Omega$ - longitude of the as-
cending node, $i$ - inclination of the orbit, $\omega$ - angular distance from perihelion to the ascending node. The corresponding formulae have the form:

$$
\begin{aligned}
& P_{1}=\cos \Omega \cos \omega-\sin \Omega \sin \omega \cos i \\
& P_{2}=\sin \Omega \cos \omega-\cos \Omega \sin \omega \cos i \\
& P_{3}=\sin \omega \sin i \\
& Q_{1}=-\cos \Omega \sin \omega-\sin \Omega 2 \cos \omega \cos i \\
& Q_{2}=-\sin \Omega \sin \omega+\cos \Omega \cos \omega \cos i \\
& Q_{3}=\cos \omega \sin i
\end{aligned}
$$

§6. The analysis of the errors resulting from the use of the Runge-Kutta-Gill, de Vogelaere, Adams-Störmer and Gautschi method was carried out by solving equations (2) by the above mentioned methods and by comparing the results with the exact Kepler solutions (3).

The difference between the exact and the approximate solutions was taken as the value of error. Since the starting points needed to begin computations in the methods of numerical integration were determined on the basis of the exact formulae (3), the deviation of the approximate solution from the exact one is due only to the use of integral formulae of the fourth order for all methods and to the summing up of all round-off errors in each step of integration. The following values of elements of a fictitious orbit were adopted for computations performed on Ural-2 digital computer:

$$
\begin{align*}
& \Omega=80^{\circ} .7 \\
& i=10^{\circ} .6 \\
& \omega=69^{\circ} .9  \tag{4}\\
& \mu=0^{\circ} .21415111 \quad\left(a=2.767 \quad \text { a. u. } .^{*}\right) \\
& e=0.1,0.3,0.6,0.9
\end{align*}
$$

The choice of the semi-major axis a was due to an analogy with computations, most frequent in astronomy, of orbits of short-period comets and planetoids semi-major axes of which are of the same order as the assumed value $a$. All the computations were performed for the four values of eccentricity $e$ given above, in order to establish the influence of the shape of the orbit upon the value of deviation. The step of integration $h$ was represented in each method by the integral parts of the solutions period $T$, the following values being adopted for all the methods:

$$
\frac{T}{100}, \frac{T}{200}, \frac{T}{400}, \frac{T}{800}
$$

[^1]The integration of equations (2) was performed in the time interval equal to ten periods of the motion of the body under consideration around the central body.
§7. The analysis of changes in time of the deviation of solutions obtained by the approximate methods from the exact solutions proves that the total error in each step of integration oscillates near the zero value as the number of steps increases. The character of these changes is presented by the diagrams in Fig. 1, which shows the dependence of the deviation upon the Kepler solutions for the four methods in question, (for example for the coordinate $x$ of the deviation; similar changes can be observed for the other coordinates) which were used for the integration of equations (2) with the step $\frac{T}{200}$, with the three different values of the eccentricity: 0.1 (fig. 1a), 0.3 (fig. 1b), 0.6 (fig. 1c) and with the orbital elements (4).

The step of integration is marked on the horizontal axis in the same scale for all three diagrams, while the deviation is marked on the vertical axis; the deviations $10^{-5}$ a. u. in diagrams a) and b), and $10^{-8}$ a. u. in diagram c) correspond to the distance between two successive scale marks for the Runge-Kutta-Gill, de Vogelaere and Adams-Storrmer methods, while the scale of $10^{-3} \mathrm{a}$. $u$. is assumed in the all diagrams for the error resulting from the Gautschi method. The conse of the given coordinate in time is presented under each of the three diagrams, with equal scales on the abscissae.

The diagrams in fig. 1 show that:

1. The period of changes of deviations of the approximate solutions from the exact solutions is the same for all the methods and equals the solution period $T$.
2. The deviation changes its sign as the first derivative of the coordinate with respect to time equals zero, that is as the corresponding component of the velocity of the body in the orbit vanishes.
3. The absolute value of the deviation is the greatest when the corresponding component of velocity have extreme values.
4. The amplitude of deviations increases with the increase of the number of the steps performed. The amplitude is different for all the methods in the corresponding periods. Different methods have different rates of the increase in the amplitude.
5. A strong dependence of the shape of the error curve and of the value of deviation on the shape of the orbit, first of all on its eccentricity, is observed. Such dependence on other orbital elements is not found.
§8. Since changes in time of the deviation of the approximate solution from the exact one are periodic it is necessary to define a definite measure of error in some time interval in order to establish criteria of the choice of the method and the proper step of integration. The maximal




Fig. 1
deviation in one full motion $T$ of the body under consideration around the central body, regardless of its sign, is adopted as the measure of error.

1. The dependence of the maximal deviation on the eccentricity of the orbit is shown in fig. 2 for all the methods. Eccentricity and the logarithm of deviation are marked on the axis of abscissae and the axis of ordinates, respectively. The large values of maximal error for large eccentricities account for the use of a logarithmic scale to determine the deviation. The diagrams a) show the course of the considered function after one solution period, the diagrams b ), after three periods and the diagrams c ),


Fig. 2
after ten periods. Every courve is plotted for the appropriate step of integration. The diagrams show that:
a) The step of integration in certain ranges of its value has a comparatively small effect on the value of maximal deviation for some eccentricities (there are points in which the curves corresponding to various steps meet). For example for the Runge-Kutta-Gill method after three periods of integration the maximal error is almost the same, for the orbit with the eccentricity of ca. 0.4 and for integration with the steps $\frac{T}{200}$,
$\frac{T}{400}$, or $\frac{T}{800}$. Similarily, when integrating the equations of motion the trajectory of which has the eccentricity of 0.3 by the de Vogelaere method, after three periods the error is nearly the same regardless of the step used: $\frac{T}{100}, \frac{T}{200}, \frac{T}{400}$, or $\frac{T}{800}$.
b) The increase of maximal deviation with respect to the eccentricity of the orbit is slower for smaller steps of integration for all the methods considered.
c) For the Adams-Störmer and Gautschi difference methods a comparatively smaller "dispersion" of curves is observed. This shows that the effect of a step of integration on the value of maximal deviation is smaller than in the remaining methods.
2. The dependence of maximal deviation of an approximate solution from the exact one upon the value of the step of integration used in computations is given in fig. 3.

Step of integration is marked on the horizontal axis and the corresponding values of deviation, in the logarithmic seale, as in the former case, are marked on the vertical axis. The diagrams in the columns a), b), c) show the course of the function after one, three and ten solution periods respectively. The considered relations for the four values of eccentricity: $0.1,0.3,0.6,0.9$, are given in four successive lines, in which the diagrams are placed. In every diagram different curves correspond to different methods. The diagrams in fig. 3 show that:
a) For small eccentricities more exact results can be obtained by using greater steps of integration $\left(\frac{T}{100}, \frac{T}{200}\right)$. This can be explained by the small effect of the cut-off error and the small accumulation of error when a small number of steps of integration is performed. On the contrary, for orbits with large eccentricities integration with smaller steps markedly reduces the maximal error because the cut-off error now becomes more important than the accumulation of error.
b) A similar dependence of maximal deviation on the step of integration is observed for the Runge-Kutta-Gill and do Vogelaere methods.
c) The greatest errors, regardless of the step of integration and the orbit shape, are yielded by the Gautschi method.
3. The diagrams in fig. 4 show the rate of the increase of the maximal deviation in time. Time is marked on the abscissa, its unit being one full solution period T. The logarithm of maximal deviation is marked on the


Fig. 3
ordinate, as before. Every diagram shows the course of the considered function for each particular method (separate curves), with a first step of integration and eccentricity of the orbit.

The similarity of these functions for the Runge-Kutta-Gill and the de Vogelaere method is interesting. As mentioned above, resemblance between these methods is also revealed by the dependence of maximal deviation on the step of integration. But as can be seen from the diagrams in fig. 1, the de Vogelaere method has a less regular course of the error curve in comparison with the Runge-Kutta-Gill method.


Fig. 4
4. The table below gives a relative estimation of the time needed for performing one step of integration by each of the methods under consideration while the corresponding time of the exact Kepler method is taken as the unit; the number of cells of digital computer operational memory that must be reserved for a given method when solving a system of equations of type ( 2 ) $(n=3)$; the number of starting points needed to begin computations by a given method:

| Method | Runge-Kutta-Gill | de Vogelaere | Adams-Störmer | Gautschi |
| :--- | :---: | :---: | :---: | :---: |
| Time | 1.66 | 0.89 | 1.00 | 0.78 |
| Number of <br> cells | 33 | 15 | 46 | 26 |
| Number of <br> starting <br> points | 1 | 1 | 6 | 3 |

The conclusions formulated above refer to the analysis of errors of the said methods of numerical integration of differential equations and their applicability to computations. These conclusions can be used as criteria facilitating the choice of the best method and the best step of integration for a given problem. For instance, when one wants to find the coordinates of a body over a small time interval as exactly as possible and the body in question has an orbit which approximates ot a circle and thus has small eccentricity, and when it is advisable to perform computations as quickly as possible using a possibly small number of cells in the operational memory of a digital computer, then considering the diagrams in figs. $2,3,4$ and the table from the point 4 and taking into account the above assumptions, one should choose the de Vogelaere method with a large step of integration e.g. $\frac{T}{100}$, as the most suitable one. If one wants to perform integration over a large time interval, e.' g. eight periods. then the Adams-Störmer difference method, with the step e. g. $\frac{T}{100}$, will be more suitable with respect to the value of the maximal error, in spite of a greater number of memory cells and a longer time needed.
§9. The shape of the curve representing the change in time of the errors yielded by each method of numerical integration of type (2) differential equations (fig. 1) suggests that it is possible to find empirically such a correction which, when adopted to the approximate colution, will make it come close as far as possible to the exact one. It is obvious that the radius vector error where the radius vector ( $r^{2}=x_{1}^{2}+x_{2}^{2}+x_{3}^{2}$ ) will change in time ${ }^{*}$, analogically as each coordinate ( $x_{1}, x_{2}, x_{3}$ ) does. The shape of the curve of these changes gives reasons for assuming the following expansion of the deviation of the radius $\Delta r$ :

$$
\begin{equation*}
\Delta r=a_{1}(t) \cos h t+a_{2}(t) \cos 2 h t+\ldots \tag{5}
\end{equation*}
$$

where $a_{1}(t) ; a_{2}(t)$, etc. are time functions, an unknown form of which should be defined beforehand. The shape of the envelope of the curves in fig. 1 seems to point to the possibility of assuming the quadratic function as the form of these relations:

$$
\begin{aligned}
& a_{1}(t)=a_{10}+a_{11} t+a_{12} t^{2} \\
& a_{2}(t)=a_{20}+a_{21} t+a_{22} t^{2}
\end{aligned}
$$

The coefficients $a_{i j}(i=1,2, \ldots ; j=0,1,2)$ are merely functions of the step of integration and of the elements characterising the shape

[^2]of the orbit namely eccentricity $e$ and semi-major axis $a$. When these three quantities are defined and the deviations $\Delta r$ of the radius vector in different moments are known one can solve equations (5), e. g., by the method of least squares in order to determine the sought coefficients. When these are known for various values of $h$, e and $a$, one can correct the radius $r$ by adding the value of the computed correction $\Delta r$ in each step of integration, considerably increasing thereby the precision of the solution.

The next paper of the present author will be concerned with finding the coefficients $a_{i j}$ for the various values of steps of integration, eccentricity and the semi-major axis of the orbit.

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[^0]:    * The paper was submitted on November 1963.

[^1]:    * Astronomical unit (a. u): the mean distance from the Earth to the Sun. It is equal to $149.55 \times 10^{6}$ kilometers.

[^2]:    * Time here means rather the number of integration steps performed than absolute time.

