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COMPARISON OF PURCELL'S METHOD WITH THE ELIMINATION METHOD OF GAUSS

by

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Резюме

В этой работе произведён анализ алгоритма Перселла и показана связь этого алгоритма с методом исключения Гаусса. Кроме того работа содержит краткое сравнение этих методов с точки зрения их практического применения для решения различных проблем линейиой алгебры, при чём особое внимание обращено на применение метода Перселла ври сичислении иа электронных цифровых машинах.

Summary

The paper presents an anlysis of Purcell's algorithm, and some relation to the elimination method of Gauss. Short comparison of the two methods is included regarding their possibilities in solving different problems of linear algebra, and particularly their use for performing calculations on electronic digital computers.

1. Purcell's Algorithm

The problem of finding the solution $X = [x_1, x_2, \ldots, x_n]$ of nonsingular system of algebraic linear equations

$$
(1.1)\qquad
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n - f_1 &= 0\\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n - f_2 &= 0\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n - f_n &= 0
\end{aligned}
$$

is equivalent*) to the problem in which we have to find vector

$$(1.2)\qquad Y = [X, 1],$$

* [1], [2].

perpendicular to every vector

$$(1.3) \qquad A_j = [a_{j1}, a_{j2}, \ldots, a_{jn}, -f_j]; \quad (j = 1, 2, \ldots, n),$$

i. e. satisfying conditions

$$(1.4) \qquad (A_j, Y) = 0; \quad (j = 1, 2, \ldots, n).$$

One of the algorithms by means of which it is possible to find such vector Y is the Purcell's algorithm. If I_s will denote a versor of s-axis of $n+1$-dimensional Cartesian space then this algorithm may be written in the abbreviated form by means of the following recurrence formulae:

$$(1.5) \qquad V_s^{(0)} = I_s; \quad (s = 1, 2, \ldots, n+1);$$

$$(1.6) \qquad c_i^{(k)} = \frac{(A_k, V_i^{(k-1)})}{(A_k, V_k^{(k-1)})};$$

$$(1.7) \qquad V_i^{(k)} = V_i^{(k-1)} - c_i^{(k)} V_k^{(k-1)}; \quad (k = 1, 2, \ldots, n; \quad i = k+1, \ldots, n+1)$$

$$(1.8) \qquad Y = V_{n+1}^{(n)}$$

on condition

$$(1.9) \qquad (A_k, V_k^{(k-1)}) \neq 0; \quad (k = 1, 2, \ldots, n).$$

The fact that vector X determined by (1.2), (1.8) is a solution of system (1.1) follows from the trivial theorem which can be formulated as follows:

THEOREM 1. *If for every* $q = 1, 2, \ldots, k (1 \leqslant k \leqslant n)$ *we have*

$$(A_q, V_q^{(q-1)}) \neq 0,$$

then for every $q = 1, 2, \ldots, k$ *and for every* $i = k+1, \ldots, n+1$ *we have*

$$(1.10) \qquad (A_q, V_i^{(k)}) = 0.$$

The proof of Theorem 1 is immediate if we consider that using (1.6) and (1.7) for every $q = 1, 2, \ldots, k$ we have

$$(1.11) \qquad (A_q, V_p^{(q)}) = 0; \quad (p = q+1, \ldots, n+1),$$

and that for every $r = k+1, \ldots, n+1$, vectors $V_r^{(k)}$ are linear combinations of vectors $V_p^{(q)}$ $(p = q+1, \ldots, n+1)$.

From Theorem 1 follows that

$$(1.12) \qquad (A_j, V_{n+1}^{(n)}) = 0; \quad (j = 1, 2, \ldots, n).$$

And still, from the construction of successive systems of vectors $V_j^{(k)}$ we conclude that the last coordinate of vector $V_{n+1}^{(n)}$ equals unity.

Therefore, vector X determined by (1.2), (1.8) is the solution of system (1.1). Table 1 gives an example of solving the system of 4 equations with two columns $F, \bar{F}$ consisting of constant terms, by means of Purcell's algorithm. The corresponding solutions $X, \bar{X}$ are in rows $V_5^{(4)}, \bar{V}_5^{(4)}$.

2. Conditions for the System to be Solvable by Means of Purcell's Algorithm

If any of conditions (1.9) is not satisfied, then the system cannot be solved using the Purcell's algorithm as the corresponding coefficient $C_i^{(k)}$ is indefinite. Now, let us formulate and prove a theorem determining the conditions sufficient for satisfying (1.9).

THEOREM 2. *For vector X determined by* (1.2), (1.8) *be a unique solution of system* (1.1) *it will be sufficient that for every* $k = 1, 2, \ldots, n$

$$D_k \neq 0 \tag{2.1}$$

where

$$D_k = \begin{vmatrix} a_{11} & a_{21} & \ldots & a_{k1} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ a_{k1} & a_{k2} & \ldots & a_{kk} \end{vmatrix} \tag{2.2}$$

Let us introduce the following denotations

$$s_{ji} = (A_j, V_i^{(i-1)}); \quad (j, i = 1, 2, \ldots, n); \tag{2.3}$$

$$V_i^{(i-1)} = [z_{1i}, z_{2i}, \ldots, z_{i-1,i}, 1, 0, \ldots, 0]. \tag{2.4}$$

Form (2.4) of vector $V_i^{(i-1)}$ follows immediately from recurrence formulae (1.5), (1.6), (1.7). For proving Theorem 2 let us first prove that the following relation is right:

$$D_k = s_{11} \cdot s_{22}, \ldots, s_{kk}; \quad (k = 1, 2, \ldots, n). \tag{2.5}$$

Indeed, if $V_{i/k}^{(i-1)}$ $(i = 1, 2, \ldots, k)$ will denote a vector of k-dimensional space, that was obtained from vector $V_i^{(i-1)}$ by rejecting last $n+1-k$ zero coordinates, and $A_{j/k}$ $(j = 1, 2, \ldots, k)$ will denote vectors $[a_{j1}, a_{j2}, \ldots, a_{jk}]$ respectively, then the equation

$$s_{ji} = (A_{j/k}, V_{i/k}^{(i-1)}); \quad (j, i = 1, 2, \ldots, k)., \tag{2.6}$$

is true.

Let us now construct matrices

$$(2.7) \qquad A^{(k)} = \begin{bmatrix} A_{1/k} \\ A_{2/k} \\ \vdots \\ A_{k/k} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{bmatrix};$$

$$(2.8) \qquad Z^{(k)} = [V_{1/k}^{(0)'}, \dots, V_{k/k}^{(k-1)'}] = \begin{bmatrix} 1 & z_{12} & \dots & z_{1k} \\ 0 & 1 & \dots & z_{2k} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

The determinant of the product of these matrices equals

$$(2.9) \qquad |A^{(k)}Z^{(k)}| = |A^{(k)}| \,.\, |Z^{(k)}| = |A^{(k)}| = D_k\,,$$

as $|Z^{(k)}| = 1$.

On the other hand, from (2.3), (2.6), (2.7), (2.8), and Theorem 1 follows

$$(2.10) \qquad S^{(k)} = A^{(k)}Z^{(k)} = \begin{bmatrix} s_{11} & 0 & \dots & 0 \\ s_{21} & s_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ s_{k1} & s_{k2} & \dots & s_{kk} \end{bmatrix}.$$

Hence

$$(2.11) \qquad |A^{(k)}Z^{(k)}| = |S^{(k)}| = s_{11} \cdot s_{22}, \dots, s_{kk}$$

as a determinant of triangular matrix. From (2.9) and (2.11) we obtain (2.5). Now, the thesis of Theorem 2 can be simply obtained from (2.5). Assumption (2.1) for $k = n$ ensures the uniqueness of solution X, and from (2.3), (2.5) follows that conditions (1.9) are satisfied. This completes the proof.

Let us still note that from (2.5) we obtain the following relation expressing dependence of coefficient s_{kk} upon minors of matrix A.

$$(2.12) \qquad s_{kk} = \frac{D_k}{D_{k-1}}\,; \quad (k = 2, 3, \dots, n).$$

To show this it will be sufficient to note that

$$s_{kk} = \frac{s_{11} \dots s_{kk}}{s_{11} \dots s_{k-1,k-1}}.$$

For $k = 1$ we obtain, by simple verification, $s_{11} = a_{11}$.

3. Relations between Purcell's method and the elimination method of Gauss

The elimination method of Gauss is strictly connected with the decomposition of a matrix into the product of two triangular matrices. If B will denote the matrix of triangular system occuring in effect of eliminating:

$$B = \begin{bmatrix} 1 & b_2^{(1)} & \dots & b_n^{(1)} \\ 0 & 1 & \dots & b_n^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix} \tag{3.1}$$

and if we assume that $D_k \neq 0$, $k = 1, 2, \dots, n$, then on the basis of the theorem of matrix decomposition into the product of two triangular matrices, there exists nonsingular left-side triangular matrix G such that

$$A = G^{-1}B. \tag{3.2}$$

Taking in (2.10) $k = n$, and $S^{(n)} = S$, $A^{(n)} = A$, $Z^{(n)} = Z$ we obtain $S = AZ$, and hence

$$A = SZ^{-1}\text{*)}. \tag{3.3}$$

Since the elements of main diagonals of matrices B and Z^{-1} are the ones, then basing on the theorem of unique decomposition of a matrix into the triangular factors, we conclude that

$$\begin{gathered} G^{-1} = S, \\ B = Z^{-1} \end{gathered} \tag{3.4}$$

Let C denote the matrix of coefficients $c_i^{(k)}$ $(k = 1, 2, \dots, n-1;$ $i = k+1, \dots, n)$

$$C = \begin{bmatrix} 1 & c_2^{(1)} & \dots & c_n^{(1)} \\ 0 & 1 & \dots & c_n^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix} \tag{3.5}$$

and let C_k denote k-row of this matrix.

We shall now prove the principal theorem, which implies a very close relation of Purcell's method to the elimination method of Gauss.

THEOREM 3. *If* $D_k \neq 0$ $(k = 1, 2, \dots, n)$, *then* $B = C = Z^{-1}$.

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*) [2].

To prove this theorem, using the second of equations (3.4), it will be satisfactory to show that $CZ = I$ where I is the unit matrix of n-degree. For simplicity of writing we take

$$W_j^{(i)} = V_{j/n}^{(i)}. \tag{3.6}$$

Now, let $Q^{(k)}$ be the matrix of n-degree, and $k-1$ of its first columns are columns consisting of zeros.

$$Q^{(k)} = [0, \ldots, 0, W_k^{(k-1)}, \ldots, W_n^{(k-1)}]. \tag{3.7}$$

We shall prove the following equation

$$Q^{(k)} W_l^{(l-1)} = W_l^{(l-1)}; \quad (k = 1, 2, \ldots, n;\ l = k, \ldots, n). \tag{3.8}$$

From (1.5), (1.6), (1.7) follows that vector $W_r^{(k-1)}$ is of the form

$$W_r^{(k-1)'} = [w_{1r}^{(k-1)}, \ldots, w_{k-1,r}^{(k-1)}, 0, \ldots, 0, 1, 0, \ldots, 0]; \quad (r = k, \ldots, n) \tag{3.9}$$

and the one is r-coordinate of this vector.

Using the same formulae, vector $W_l^{(l-1)}$ $(l \geqslant k)$ can be expressed by means of the following linear combination of vectors $W_k^{(k-1)}, \ldots, W_l^{(k-1)}$.

$$W_l^{(l-1)} = d_k W_k^{(k-1)} + d_{k+1} W_{k+1}^{(k-1)} + \ldots + d_l W_l^{(k-1)}, \tag{3.10}$$

where, using (3.9), we obtain

$$\begin{aligned} d_q &= w_{ql}^{(l-1)} = z_{ql} \quad (q = k, \ldots, l-1), \\ d_l &= 1. \end{aligned} \tag{3.11}$$

Now, if the right side of (3.10) will be expressed in form of the product of two matrices, and using (3.11), we obtain (3.8).

Next, we shall prove Theorem 3. The product of matrices C and Z is of the form

$$CZ = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_3 \end{bmatrix} \cdot [V_{1/n}^{(0)'}, \ldots, V_{n/n}^{(n-1)'}] = \begin{bmatrix} C_1 W_1^{(0)}, C_1 W_2^{(1)}, \ldots C_1 W_n^{(n-1)} \\ C_2 W_1^{(0)}, C_2 W_2^{(1)}, \ldots C_2 W_n^{(n-1)} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ C_n W_1^{(0)}, C_n W_2^{(1)}, \ldots C_n W_n^{(n-1)} \end{bmatrix}. \tag{3.12}$$

We shall show that

$$C_k W_l^{(l-1)} = \begin{vmatrix} 0 & \text{for} & k \neq l, \\ 1 & \text{for} & k = l. \end{vmatrix} \tag{3.13}$$

For this we readily see that

$$C_k = \frac{1}{s_{kk}} A_{k/n} Q^{(k)}, \tag{3.14}$$

and hence

$$C_k W_l^{(l-1)} = \frac{1}{s_{kk}} A_{k/n} Q^{(k)} W_l^{(l-1)}. \tag{3.15}$$

If $k > l$ then obviously $Q^{(k)} W_l^{(l-1)} = 0$. If $k \leqslant l$ then from (3.8) follows that

$$C_k W_l^{(l-1)} = \frac{1}{s_{kk}} A_{k/n} W_l^{(l-1)}. \tag{3.16}$$

However, from Theorem 1 and (2.6) follows

$$A_{k/n} W_l^{(l-1)} = A_{k/n} V_{l/n}^{(l-1)'} = \begin{cases} s_{kk} & \text{for} \quad k = l \\ 0 & \text{for} \quad k < l \end{cases} \tag{3.17}$$

Therefore (3.13) is true, what completes the proof.

Comparing (3.3) and (3.4) with the thesis of Theorem 3 we obtain the following decomposition of matrix A into the triangular factors

$$A = SC, \tag{3.18}$$

where both matrix S and matrix C can be obtained in the process for solving the system using the Purcell's algorithm. Theorem 3 determines close interconnections between the Gauss's and Purcell's method. Moreover, Theorem 3 gives the geometrical sense to the coefficients of triangular matrix obtained in effect of elimination.

4. Choice of Main Element in Purcell's Method

While solving the system of equations by means of Purcell's algorithm it may occur that although the system is nonsingular, not all conditions (1.9) must be satisfied. In the elimination method such cases can be avoided by introducing a so called choice of main element. In what follows the suggestion is made to introduce analogous choice of the main element in the Purcell's method.

Let vectors $V_k^{(k-1)}$ be named the main vectors. In principle, it is of no difference which one of vectors $V_i^{(k-1)}$ in k-step will be the main vector provided that the main vector will satisfy the corresponding condition of (1.9). It is also quite indifferent in what order the equations are arranged in the system (1.1).

We can establish then the following principle in chosing maximum coefficient $s_{p_k q_k}$ appearing in denominator (1.6):

$$|s_{p_k q_k}| = |(A_{p_k}, V_{q_k}^{(k-1)})| = \max_{j,i}\{|(A_j, V_i^{(k-1)})|\} \quad (i \neq n+1), \tag{4.1}$$

where j belongs to the set of indices of those equations which did not appear in the procedure yet, and i belongs to the set of indices of those vectors $V_i^{(1)}$ which were not yet eliminated. This corresponds to so called full choice of a main element in the Gauss's method. It is, however, very much time consuming, and using it in practice would be unprofitable. And in the Purcell's method we can successfully use the choice of a main element, corresponding to the choice of a main element from one row only in the elimination method. This element is to be found according to the following rule.

$$(4.2)\qquad |s_{kq_k}| = |(A_k, V_{q_k}^{(k-1)})| = \max_i\{|s_{ki}|\} \quad (i = n+1).$$

If the system is nonsingular, so on this change of the algorithm always exists index q_k such that the appropriate condition (1.9) is satisfied. This follows through Theorem 3 from the appropriate analysis of the elimination method. The above ensures that every nonsingular system is solvable by means of Purcell's algorithm, and makes higher a degree of stability of performed calculations.

5. Use of Purcell's Algorithm

Possibility of chosing the main element in Purcell's algorithm makes this algorithm be, regarding stability of calculations, almost equivalent to the elimination method of Gauss. A number of arithmetic operations, i. e. multiplications, divisions, and additions are in both the methods identical and equal $\frac{n}{3}(n^2+3n-1)$, $\frac{1}{6}n(n-1)(2n+5)$ respectively.

The Purcell's algorithm has, however, some advantages not to be found in the elimination method. These are:

1) uniformity in proceeding — contrary to the elimination method which consists of two stages with different procedures each, i. e. reducing a system to the triangular system, and so called recurrence procedure;

2) in the given step k, only one row A_k of the augmented matrix of the system is required for calculations.

These advantages are of importance with regard to listing programmes for electronic digital computers, and maximum use of the computer stores. If in every step of the algorithm only one row will be entered into the machine store, then the maximum number of working places in the machine store is of the range $\frac{n^2}{4}+n$.

Table 1

	Matrix				constant terms		S_{ki}	$C_i^{(k)}$
					$-F$	$-\bar{F}$		
A_1	5	7	6	5	-23	-23.1		
A_2	7	10	8	7	-32	-31.9		
A_3	6	8	10	9	-33	-32.9		
A_4	5	7	9	10	-31	-31.1		
$V_1^{(0)}$	1	0	0	0	0		5	1
$V_2^{(0)}$	0	1	0	0	0		7	1.4
$V_3^{(0)}$	0	0	1	0	0		6	1.2
$V_4^{(0)}$	0	0	0	1	0		5	1
$V_5^{(0)}$	0	0	0	0	1		-23	-4.6
$\bar{V}_5^{(0)}$	0	0	0	0	1		-23.1	-4.62
$V_2^{(1)}$	-1.4	1	0	0	0		0.2	1
$V_3^{(1)}$	-12	0	1	0	0		-0.4	-2
$V_4^{(1)}$	-1	0	0	1	0		0	0
$V_5^{(1)}$	4.6	0	0	0	1		0.2	1
$\bar{V}_5^{(1)}$	4.62	0	0	0	1		0.44	2.2
$V_3^{(2)}$	-4	2	1	0	0		2	1
$V_4^{(2)}$	-1	0	0	1	0		3	1.5
$V_5^{(2)}$	6	-1	0	0	1		-5	-2.5
$\bar{V}_5^{(2)}$	7.7	-2.2	0	0	1		-4.3	-2.15
$V_4^{(3)}$	5	-3	-1.5	1	0		0.5	1
$V_5^{(3)}$	-4	4	2.5	0	1		-0.5	-1
$\bar{V}_5^{(3)}$	-0.9	2.1	2.15	0	1		-1.55	-3.1
$V_5^{(4)}$	1	1	1	1	1			
$\bar{V}_5^{(4)}$	14.6	-7.2	-2.5	3.1	1			

In the elimination method the corresponding number is $n(n+1)$. This is of importance particularly in solving systems with many vectors of constant terms. Problems of this kind can be solved by introducing to every column consisting of constant terms the corresponding vector $V_{n+1}^{(0)}$ (see Table 1). In this connection, the Purcell's algorithm can also be used for inversion of matrices. From (2.5) follows the fashion of calculating both determinants and main minors of matrices.

Moreover, use of (3.18) gives the possibility of decomposing a matrix into the product of two triangular matrices. The two methods are, in principle, equivalent, and differ only in proceeding. This gives in effect more variants in possibilities of these methods.

If in (1.5) we take as $V_i^{(0)}$ $(i = 1, 2, \ldots, n+1)$ any linearly independent system of vectors, then more general method will be obtained, particular cases of which will be the Purcell's algorithm and, in some sense, the elimination algorithm of Gauss.

Example of solving the system with two vectors of constant terms by means of Purcell's algorithm.

BIBLIOGRAPHY

[1] Journal of Mathematics and Physics 32, 1953. Ewerett W. Purcell, *The Vector Method of Solving Simultaneous Linear Equations*,

[2] D. K. Faddeev, V. N. Faddeeva, *Vyčislitel'nye metody linejnoj algebry*, 1960.

[3] G. E. Forsythe, *Contemporary State of Numerical Analysis*, 1958.

[4] Tomasz Pietrzykowski, *Projection Method*, Prace Zakładu Aparatów Matematycznych PAN, A8, 1960.